Problem 1

Z_i-s are independent and standard normal

a) i) $P(2Z_1 - 3Z_2 \leq 3) = P\left(Z \leq \frac{3}{\sqrt{13}}\right) = 0.797\ldots$

$N(0, 4+9=13)$

ii) $\begin{pmatrix} U \\ V \end{pmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix} \vec{Z}$

$$E\begin{pmatrix} U \\ V \end{pmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$Cov\begin{pmatrix} U \\ V \end{pmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix} I \begin{bmatrix} 1 & 3 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}$$

$$\begin{pmatrix} U \\ V \end{pmatrix} \sim MVN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}\right)$$

iii) $\vec{X} = \begin{pmatrix} Z_1 + Z_3 \\ Z_1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \vec{Z}$

$$\mu = \vec{0} \qquad \Sigma = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$X_1 | X_2 = z_1 \sim N(0 + (z_1 - 0), 2 - 1)$$

$$N(z_1, 1)$$

b) $\vec{Y} = \begin{bmatrix} \frac{2}{3} & 0 & \frac{\sqrt{2}}{3} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{3} & 0 & \frac{1}{3} \end{bmatrix} \vec{Z}$

↑ A is symmetric

$AA = \begin{bmatrix} \frac{2}{3} & 0 & \sqrt{2}/3 \\ 0 & 1 & 0 \\ \sqrt{2}/3 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 2/3 & 0 & \sqrt{2}/3 \\ 0 & 1 & 0 \\ \sqrt{2}/3 & 0 & 1/3 \end{bmatrix}$

$= \begin{bmatrix} \frac{4}{9}+\frac{2}{9} & 0 & \frac{2\sqrt{2}}{9}+\frac{\sqrt{2}}{9} \\ 0 & 1 & 0 \\ \frac{2\sqrt{2}}{9}+\frac{\sqrt{2}}{9} & 0 & \frac{2}{9}+\frac{1}{9} \end{bmatrix} = \begin{bmatrix} \frac{6}{9} & 0 & \frac{3\sqrt{2}}{9} \\ 0 & 1 & 0 \\ \frac{3\sqrt{2}}{9} & 0 & 3/9 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & 0 & \frac{\sqrt{2}}{3} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{3} & 0 & \frac{1}{3} \end{bmatrix}$

A is idempotent

$\vec{Y}^+\vec{Y} = (A\vec{Z})^T A\vec{Z} = \vec{Z}^T A^T A\vec{Z} = \vec{Z}^T AA\vec{Z}$

$= \vec{Z}^T A\vec{Z}$

Since $\vec{Z} \sim MVN(\vec{0}, I)$ then ← formula sheet

$\vec{Z}^T A\vec{Z} \sim \chi^2(r)$ $\quad r = \text{rank}(A) = \text{tr}(A)$

$= \frac{2}{3} + 1 + \frac{1}{3} = \underline{\underline{2}}$

Problem 2

a) $E(\bar{Y}) = E\left(\frac{1}{n}\vec{1}^T\vec{Y}\right) = \frac{1}{n}\vec{1}^T\vec{1}\mu = \frac{n}{n}\mu = \mu$

$Var(\bar{Y}) = Var\left(\frac{1}{n}\vec{1}^T\vec{Y}\right) = \frac{1}{n^2}\vec{1}^T\sigma^2 I\,\vec{1}$

$= \frac{\sigma^2}{n^2}\vec{1}^T\vec{1} = \frac{n}{n^2}\sigma^2 = \frac{\sigma^2}{n}$

$E(S^2) = \frac{1}{n-1} \cdot \left(\text{tr}\left(I - \frac{1}{n}J \cdot \sigma^2 I\right) + \vec{1}^T\mu\,(I - \frac{1}{n}J)\vec{1}\mu\right)$ (trace)

$= \frac{\sigma^2}{n-1}\underbrace{\text{tr}\left(I - \frac{1}{n}J\right)}_{n - \frac{1}{n}n = n-1} + \frac{\mu^2}{n-1}\underbrace{\vec{1}^T(I - \frac{1}{n}J)\vec{1}}_{}$

$J\vec{1} = n\vec{1}$

$\frac{1}{n}J\vec{1} = \vec{1}$

$\vec{1}^T\vec{1} - \vec{1}^T\vec{1} = 0$

$= \sigma^2$

$Var(S^2) = \left(\frac{1}{n-1}\right)^2\left[2\,\text{tr}\left[\underbrace{\left((I - \frac{1}{n}J)\sigma^2 I\right)^2}_{\sigma^4(I - \frac{1}{n}J)}\right] + 4\mu^2\vec{1}^T(1 - \frac{1}{n})\sigma^2 I(I - \frac{1}{n})\vec{1}\right]$

$= \left(\frac{1}{n-1}\right)^2\left[2\sigma^4(n-1) + 4\mu^2\sigma^2\underbrace{\vec{1}^T(I - \frac{1}{n}J)\vec{1}^T}_{=0}\right]$

$= \frac{2\sigma^4}{n-1}$

b) S^2 and $\bar{y}$ are independent

Linear ($\bar{y}$) and quadratic (S^2) forms

independent if $B \Sigma A = 0$

$\vec{1}^T$ ↗ ↑ ↑ $I - \frac{1}{n} J$

$\sigma^2 I$

$$B \Sigma A = \sigma^2 \vec{1}^T \left(1 - \frac{1}{n} J\right) = \vec{1}^T - \frac{1}{n} n \vec{1}^T$$

$$= \vec{1}^T - 1^T = \vec{0}.$$

Problem 3

$n = 100$

Y: productivity score

a) $\vec{Y} = X\vec{\beta} + \vec{\varepsilon}$

↑ intercept, training, experience, dep Sales, dep IT

length = 5

$X_{100 \times 5}$ — first column: $\vec{1}$

sec column : 0 or 1 for training

3rd col : years of exp.

4th col : 0/1 if Sales

5th col : 0/1 if IT

each row has numbers cor. to one employee.

b) $\hat{E(y)} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \hat{\beta}_3 x_3 + \hat{\beta}_4 x_4$

x_1: ↑ 0, x_2: ↑ 3, x_3: ↑ 0, x_4: ↑ 0

$= 47.0892 + 1.9876 \cdot 3 = 53.052$

c) $dep_{IT} = 8.0761$

prod score of employee in IT is on av. 8.076 higher than in HR, if experience and training equal.

The hypothesis $H_0: \beta_4 = 0$ vs $H_1: \beta_4 \neq 0$ and the test used is a t-test.

p-value 0.07.. means that we do not reject the null, no significant difference between HR and Sales in productivity scores.

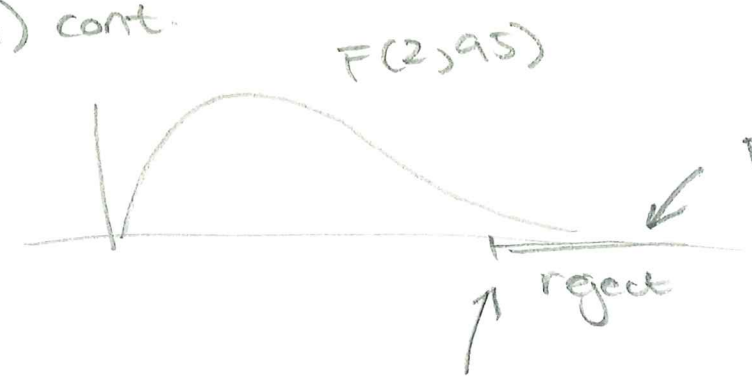
d) $\vec{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \leftarrow \text{sales} \\ \beta_4 \leftarrow \text{IT} \end{pmatrix}$ } if both are zero then all departments are the same → no effect of department.

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad C\vec{\beta} = \begin{pmatrix} \beta_3 \\ \beta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Test stat F has central F distn under the null, with dof, $q = 2$ and $n - (\text{null}) = 95$.

($n = 100$, null $= 5$)

d) cont.



$F_{obs} = 14.2999$
reject → department is relevant.

critical value at $\alpha = 0.01$ ≈ 4.8

Problem 4

a) ? Std. Error 0.3474

t-value $-1.2155 / 0.3474 = -3.4988\ldots$

dof 3 and $16-4=12$

b) (i) B low level (-1)

A low level : $-2.3457 + (-1)(-1)\,2.9075$

A high level : $+2.3457 + (-1)(+1)\,2.9075$

Est difference : Contribution to $\hat{y}$ A low level : 0.5618
Contribution to $\hat{y}$ A high level : -0.5618

$\rightarrow 2 \cdot 0.5618 = 1.1236$

A from low to high : decrease $\hat{y}$ by 1.1236

A from high to low : increase $\hat{y}$ by 1.1236

(when B low level)

(ii) B high level

A low level : $-2.3457 + (+1)(-1)\,2.9075 = -5.2532$

A high level : $2.3457 + 2.9075 = +5.2532$

10.5064

A from low to high : increase $\hat{y}$ by 10.5064

high to low : decreaces $\hat{y}$ by 10.5064

(when B high level)