PLAN

8:15 – 8:20

8:35 – 8:40 Solutions 3.6.1

9:00 – 9:15 Break

9:15 – 9:20 Solutions 3.6.2

9:50 – 10:00 Solovtions to 3.20, 3.21

Write down ntnu-username(s) at sheet and hand in.

Some key results chap 3.6

$\vec{Y}$ random n-vector, mean $\vec{\mu}$, cov. matrix Σ $(n \times n)$

$\vec{a}, \vec{b}$ n-vectors
A $p \times n$ matrix
$\vec{c}$ p-vector
} constant

Linear functions:

$\vec{a}^T \vec{Y} = V$, $\vec{b}^T \vec{Y} = W$, $\vec{a}^T \vec{Y} + \vec{b}$ (univariate)

$A\vec{Y} = \vec{U}$ ← random p-vector

$A\vec{Y} + \vec{c}$

Expectation:

$E(\vec{a}^T \vec{Y}) = E(V) = \vec{a}^T \vec{\mu}$

$E(\vec{a}^T \vec{Y} + \vec{b}) = \vec{a}^T \vec{\mu} + \vec{b}$

Variance:

$Var(V) = \vec{a}^T \Sigma \vec{a}$ "$Cov(\vec{a}^T \vec{Y})$"

Covariance:

$cov(V, W) = cov(\vec{a}^T \vec{Y}, \vec{b}^T \vec{Y}) = \vec{a}^T \Sigma \vec{b}$

$cov(\vec{U}) = cov(A\vec{Y}) = A\Sigma A^T$ $(p \times p)$

$cov(\vec{U} + \vec{c}) = cov(\vec{U}) = A\Sigma A^T$.

3.6.1

a)(i) $\vec{a} = \frac{1}{n}\vec{j}$ $\quad \vec{a}^T\vec{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i = \bar{Y}$ Average of n random variables

(ii) $E(\vec{a}^T\vec{Y}) = \vec{a}^T\vec{\mu} = \frac{1}{n}\sum_{i=1}^{n}\mu_i$ Average of n mean values

(iii) $\frac{1}{n}\sum_{i=1}^{n}\mu = \frac{n\mu}{n} = \mu$ $\quad E(\bar{Y}) = \mu$

b) $\vec{Z} = A\vec{Y} = \begin{pmatrix} 3Y_1 + 3Y_2 \\ 4Y_2 \\ 5Y_1 + Y_2 \end{pmatrix}$ ← each element is a linear combination of Ys.

$$E(\vec{Z}) = E(A\vec{Y}) = A\vec{\mu} = \begin{pmatrix} 3\mu_1 + 3\mu_2 \\ 4\mu_2 \\ 5\mu_1 + \mu_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix}$$

c) $A\vec{Y} = (I - \frac{1}{n}J)\vec{Y}$

$= \vec{Y} - \bar{Y}\vec{j}$

random n-vector with element $Y_i - \bar{Y}$.

$$E(A\vec{Y}) = A\vec{\mu} = \mu(I - \tfrac{1}{n}J)\vec{j} = \mu(\vec{j} - \tfrac{1}{n}n\vec{j}) = \vec{0}$$

3.6.2

a) $\sigma_{11} = \sigma_1^2 = Var(Y_1)$ $\sigma_{12} = cov(Y_1, Y_2)$ etc...

b)(i) $Var(\vec{j}^T \vec{Y}) = \vec{j}^T \Sigma \vec{j} = (\sigma_{11} + \sigma_{21}, \sigma_{12} + \sigma_{22}) \vec{j}$

$= \sigma_{11} + \sigma_{21} + \sigma_{12} + \sigma_{22} = \sigma_{11} + \sigma_{22} + 2\sigma_{12}$

$= Var(Y_1) + Var(Y_2) + 2 cov(Y_1, Y_2)$.

(ii) $Var(\underbrace{\frac{1}{n}\vec{j}^T \vec{Y}}_{\bar{Y}}) = \frac{1}{n^2} \vec{j}^T \Sigma \vec{j}$

$= \frac{\sigma^2}{n^2} \vec{j}^T \vec{j} = \frac{\sigma^2}{n^2} n = \frac{\sigma^2}{n}$

c) $E(\vec{Z}) = \Sigma^{-1/2} \underbrace{E(\vec{Y} - \vec{\mu})}_{= \vec{0}} = \vec{0}$

$Cov(\vec{Z}) = Cov(\Sigma^{-1/2} \vec{Y})$

$= \Sigma^{-1/2} \Sigma (\Sigma^{-1/2})^T$

$= (C D^{-1/2} C^T)(C D C^T)(C D^{-1/2} C^T)$

$= C D^{-1/2} \underbrace{C^T C}_{I} D \underbrace{C^T C}_{I} D^{-1/2} C^T$

$= C \underbrace{D^{-1/2} D D^{-1/2}}_{I} C^T = C C^T = I$.

3.20 a) $z = \vec{a}^T \vec{y}$ with $\vec{a} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$$E(z) = \vec{a}^T \vec{\mu} = 2 + 3 + 3 = 8$$

$$\text{Var}(z) = \vec{a}^T \Sigma \vec{a}$$

$$= \vec{a}^T \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = -2 + 3 + 1 = 2.$$

b) $$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \end{bmatrix}$$

$$\vec{z} = A\vec{y}.$$

$$E(\vec{z}) = A\vec{\mu} = \begin{pmatrix} 1 - 1 + 3 \\ 3 - 1 - 6 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$\text{Cov}(\vec{z}) = A \Sigma A^T$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 3 & 10 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 6 & -1 \\ 13 & -17 \end{bmatrix} = \begin{bmatrix} 2+6+13 & 4-1-17 \\ 6+6-26 & 12-1+34 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & -14 \\ -14 & 45 \end{bmatrix}$$

$$\rho_{12} = \frac{-14}{\sqrt{21 \cdot 45}} = -0.455$$

3.21 a) $\vec{w} = B\vec{v}$ $\quad B = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 1 & 1 & 2 \end{bmatrix}$

$$E(\vec{w}) = B\vec{\mu} = \begin{pmatrix} 2+1+3 \\ 1-2-9 \\ 1-1+6 \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \\ 6 \end{pmatrix}$$

$$\text{Cov}(\vec{w}) = B\Sigma B^T = \begin{bmatrix} 6 & -14 & 18 \\ -14 & 67 & -49 \\ 18 & -49 & 57 \end{bmatrix}$$

b) $\text{Cov}(\vec{Z}, \vec{W}) = \text{Cov}(A\vec{Y}, B\vec{Y}) = A\Sigma B^T$

thm 3.6 d)

$$= \begin{bmatrix} 11 & -25 & 34 \\ -8 & 53 & -31 \end{bmatrix}$$

$\text{Cov}(Z_1, W_2)$

$\text{Cov}(Z_1, W_3)$

$\text{Cov}(Z_1, W_1)$

$$\text{Cov}\begin{pmatrix} \vec{Z} \\ \vec{W} \end{pmatrix} = \begin{bmatrix} \text{Cov}(\vec{Z}) & \text{Cov}(\vec{Z}, \vec{W}) \\ (2\times 2) & (2\times 3) \\ \text{Cov}(\vec{W}, \vec{Z}) & \text{Cov}(\vec{W}) \\ (3\times 2) & (3\times 3) \end{bmatrix}$$

5×5.

$\text{Cov}(\vec{Z}, \vec{W})^T$