8:35 4.1b

8:55 4.4c.

9:35 4.4d

9:50 Exam (?)

Properties, $\vec{Y} \sim MVN(\vec{\mu}, \Sigma)$

- $\vec{a}^T \vec{Y} \sim N(\vec{a}^T \vec{\mu}, \vec{a}^T \Sigma \vec{a})$
- $A\vec{Y} \sim MVN(A\vec{\mu}, A\Sigma A^T)$

$$\vec{Y} = \begin{pmatrix} \vec{Y}_1 \\ \vec{Y}_2 \end{pmatrix}$$

- if $cov(\vec{Y}_1, \vec{Y}_2) = 0 \Leftrightarrow \vec{Y}_1, \vec{Y}_2$ independent
- $\vec{Y}_1 | \vec{Y}_2 = \vec{y}_2 \sim MVN(\vec{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\vec{y} - \vec{\mu}_Y), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$

4.16 a) $\begin{pmatrix} Y_1 \\ Y_3 \end{pmatrix} \sim MVN_2\left(\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 & -1 \\ -1 & 5 \end{pmatrix} \right)$

b) $Y_2 \sim N(2, 6)$

c) $Z = \vec{a}^T \vec{Y}$ by thm 4.1, normal

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}$$

$E(Z) = \vec{a}^T \vec{\mu} = -4 \quad Var(Z) = \vec{a}^T \Sigma \vec{a} = 79.$

$Z \sim N(-4, 79)$

d) $\vec{Z} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & -1 & -1 \\ -3 & 1 & 2 & -2 \end{bmatrix}}_{A} \vec{Y}$

$rank(A) = 2$ OK.

$\vec{Z} \sim MVN_2\left(\begin{pmatrix} 2 \\ 9 \end{pmatrix}, \begin{bmatrix} 11 & -6 \\ -6 & 154 \end{bmatrix} \right)$

e) $\vec{Y} = \begin{pmatrix} \vec{Y}_A \\ \vec{Y}_B \end{pmatrix} \qquad \vec{Y}_A = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$

$$E(\vec{Y}_A | \vec{y}_B) = \vec{\mu}_A + \Sigma_{AB} \Sigma_{BB}^{-1} (\vec{y}_B - \vec{\mu}_B)$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 5 & -4 \\ -4 & 4 \end{bmatrix}^{-1} \left(\vec{y}_B - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{bmatrix} 1 & 1.5 \\ 1 & 0.5 \end{bmatrix} \vec{y}_B$$

$$= \begin{pmatrix} 1 + y_3 - 1.5y_4 \\ y_3 + 0.5y_4 \end{pmatrix}$$

$$\text{Cov}(\vec{Y}_A | \vec{y}_B) = \Sigma_{AA} - \Sigma_{AB} \Sigma_{BB}^{-1} \Sigma_{BA}$$

$$= \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$$

Thm 4.4c

$$\vec{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \qquad \vec{Y} \sim MVN_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{12} \end{bmatrix} \right)$$

if $M_{\vec{Y}}(\vec{t}) = M_{Y_1}(t_1) M_{Y_2}(t_2)$ then

Y_1, Y_2 independent.

$$M_{Y_1}(t_1) = e^{\mu_1 t_1 + \sigma_1 t_1^2/2} \qquad M_{Y_2}(t_2) = e^{\mu_2 t_2 + \sigma_2 t_2^2/2}$$

$$M_{\vec{Y}}(\vec{t}) = e^{\vec{t}^T \vec{\mu} + \vec{t}^T \Sigma \vec{t}/2}$$

$\vec{t}^T \vec{\mu}$: $t_1\mu_1 + t_2\mu_2$ ok

$\vec{t}^T \Sigma \vec{t}$: $(t_1\ t_2) \begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{bmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$

$= (t_1\ t_2) \begin{pmatrix} t_1\sigma_{11} \\ t_2\sigma_{22} \end{pmatrix} = t_1^2\sigma_{11} + t_2^2\sigma_{22}$

$$= e^{t_1\mu_1 + t_2\mu_2 + t_1^2\sigma_{11}/2 + t_2^2\sigma_{22}/2}$$

$$= M_{Y_1}(t_1)\, M_{Y_2}(t_2) \quad \blacksquare$$

Thm 4.4 d)

a) $E(Y|X) = \mu_Y + \sigma_{YX}\sigma_{XX}^{-1}(X - \mu_X)$

$$= \underbrace{\mu_Y - \frac{\sigma_{YX}}{\sigma_{XX}}\mu_X}_{\beta_0} + \underbrace{\frac{\sigma_{YX}}{\sigma_{XX}}}_{\beta_1} X$$

$$\text{Var}(Y|X) = \sigma_{YY} - \sigma_{YX}\sigma_{XX}^{-1}\sigma_{XY}$$

$$= \underbrace{\sigma_{YY} - \frac{\sigma_{YX}^2}{\sigma_{XX}}}_{\sigma^2}$$

b) No, in the figure, $\text{Var}(Y|X)$ is dependent on x.

Exam 2024

$$\vec{Y} = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \vec{X}$$

$$\vec{Y} \sim MVN_2\left(\begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{bmatrix} 11 & -05 \\ -0.5 & 4 \end{bmatrix}\right)$$

$$Z = X_1 + a X_2$$

$$Cov(Z, Y_2) = (1 \quad a)\begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= (1 \quad a)\begin{pmatrix} 1.5 \\ 2.5 \end{pmatrix} = 1.5 + 2.5a = 0$$

$$a = -\frac{1.5}{2.5} = -0.6.$$

Exam 2011

a) $\vec{Y} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \vec{X}$

$\vec{Y} \sim MVN_2$

$$E(\vec{Y}) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mu \\ k\mu \end{pmatrix}$$

$$= \begin{pmatrix} \mu(k+1) \\ \mu \end{pmatrix}$$

$$\Sigma = \begin{bmatrix} \sigma^2 & \rho\sqrt{\sigma^2 k^2}\sqrt{\sigma^2} \\ \rho\sqrt{\sigma^2 k^2}\sqrt{\sigma^2} & \sigma^2 k^2 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & \rho k \\ \rho k & k^2 \end{bmatrix}$$

$$Cov(\vec{Y}) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \sigma^2 \begin{pmatrix} 1 & \rho k \\ \rho k & k^2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \sigma^2 \begin{pmatrix} 1+\rho k & 1 \\ \rho k + k^2 & \rho k \end{pmatrix}$$

$$= \sigma^2 \begin{pmatrix} 1+2\rho k + k^2 & 1+\rho k \\ 1+\rho k & 1 \end{pmatrix}$$

b) $E(\underbrace{X_1 + X_2}_{Y_1} | \underbrace{X_1 = x_1}_{Y_2})$

$$\underbrace{\mu(k+1)}_{\mu_1} + \underbrace{(1+\rho k)\sigma^2}_{\Sigma_{12}} \cdot \underbrace{\frac{1}{\sigma^2}}_{\Sigma_{22}^{-1}} \left(\underbrace{x_1}_{y_2} - \underbrace{\mu}_{\mu_2} \right)$$

$$= \mu(k+1) + (1+\rho k)(x_1 - \mu)$$

Student has $x_1 = \mu + 1$ $\quad k = 4, \rho = \frac{1}{2}$

$$\rightarrow \mu \underbrace{(k+1)}_{5} + \underbrace{(1+\rho k)}_{1 + \frac{1}{2} \cdot 4 \; = \; 3} (\mu + 1 - \mu)$$

$$5\mu + 3$$