Exam 2014, Problem 1.

$$\vec{X} \sim MVN_3(\vec{j}, I)$$

$$A = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \qquad \vec{Y} = A\vec{X}$$

NB $A = \frac{1}{3}(3I - J) = (I - \frac{1}{3}J) = (I - \frac{1}{n}J)$, $n=3$

a) $E(\vec{Y}) = A\vec{j} = \vec{0}$

$$Cov(\vec{Y}) = AIA^T = (I - \tfrac{1}{n}J)(I - \tfrac{1}{n}J) = (I - \tfrac{1}{n}J)$$

(symmetric, idempotent.)

X_1 and X_2 have $Cov(X_1, X_2) = 0$. Since $\vec{X}$ is multivariate normal, cov 0 implies independence. Y_1, Y_2 not independent, $Cov(Y_1, Y_2) = -1/3$

$$E(X^TAX) = tr(A\Sigma) + \vec{\mu}^TA\vec{\mu}$$
$$= tr((I - \tfrac{1}{n}J) \cdot I) + \vec{j}^T(I - \tfrac{1}{n}J)\vec{j}$$
$$= n - 1 + 0$$
$$= n - 1$$

b) Since A is symmetric and idempotent, $tr(A) = rank(A)$. $tr(A) = n-1$ so $rank(A) = n-1$.

$\vec{X}^TA\vec{X}$ is a quadratic form, by corollary 2 to theorem 5.5

$$\vec{X}^TA\vec{X} \sim \chi^2(n-1, \lambda) \qquad \lambda = \frac{1}{2}\vec{j}^T(1 - \tfrac{1}{n}J)\vec{j} = 0.$$

So $\vec{X}^T A \vec{X}$ is $\chi^2(n-1)$. $\quad n-1=2$

$P(\vec{X}^T A \vec{X} \le 6) = 0.95$ (use R or Blue Booklet)

Noncentral χ^2

$U \sim \chi^2(n, \lambda) \quad U = \vec{Y}^T \vec{Y} \quad \vec{Y} \sim MVN_n(\vec{\mu}, I)$

$\lambda = \frac{1}{2}\vec{\mu}^T\vec{\mu}$

$E(U) = tr(I) + \vec{\mu}^T\vec{\mu} = n + \vec{\mu}^T\vec{\mu} = n + 2\lambda$.

↑ thm 5.2a

$Var(U) = 2\,tr(I) + 4\vec{\mu}^T\vec{\mu} = 2n + 8\lambda$.

Book

5.25 $\quad u = \dfrac{n\bar{Y}^2}{\sum_{i=1}^n (Y_i - \bar{Y})^2/n-1} = \dfrac{\sum_{i=1}^n \bar{Y}^2}{\sum_{i=1}^n (Y_i-\bar{Y})^2/(n-1)}$

$= \dfrac{\vec{Y}^T(\frac{1}{n}J)Y \cdot 1/\sigma^2}{(\vec{Y}^T(I-\frac{1}{n}J)\vec{Y}/(n-1)) \cdot \frac{1}{\sigma^2}}$

$\frac{1}{n}J$ idempotent, rank 1

$(I-\frac{1}{n}J)$ idempotent, rank $n-1$

$\frac{1}{n}J(I-\frac{1}{n}J) = \frac{1}{n}(J-\frac{1}{n}nJ) = 0 \rightarrow$ the two quadratic forms are independent.

$U = \frac{1}{\sigma^2}\vec{Y}^T(\frac{1}{n}J)\vec{Y} \sim \chi^2(1, \lambda)$

Corollary 2, p118 →

$\lambda = \dfrac{\vec{\mu}^T \frac{1}{n}J\vec{\mu}}{2\sigma^2} = \dfrac{n\bar{\mu}^2}{2\sigma^2} = \dfrac{n\mu^2}{2\sigma^2}$

$$V = \frac{1}{\sigma^2} \vec{Y}^T \left(I - \frac{1}{n} J\right) \vec{Y} \sim \chi^2(n-1, \lambda_V)$$

$$\lambda_V = \frac{\mu^2}{2\sigma^2} \vec{j}^T \left(I - \frac{1}{n} J\right) \vec{j} = 0.$$

$$V \sim \chi^2(n-1)$$

$$\frac{U/1}{V/(n-1)} \quad \text{and} \quad U, V \text{ independent}$$

then $u \sim F(1, n-1, \lambda)$ (non-central F)

if $H_0: \mu = 0$ is true then $\lambda = 0$ and $u \sim F(1, n-1)$ (central F)

5.30

$\vec{Y} \sim MVN(\vec{\mu}, \sigma^2 I)$

a) Cor. 2: $\frac{1}{\sigma^2}\vec{Y}^T A\vec{Y} \sim \chi^2(\text{rank}(A), \frac{1}{2\sigma^2}\vec{\mu}^T A\vec{\mu})$

$A = \frac{1}{3}(3I - J) = (I - \frac{1}{n}J) \quad n=3$

$\text{rank}(A) = n-1 = 2$

A is symmetric and idempotent

b) $\vec{Y}^T A\vec{Y}$ and $B\vec{Y}$ independent if

$B\Sigma A = 0$

$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} I (I - \frac{1}{n}J) = B - \frac{1}{n}BJ$

$BJ = \begin{pmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow B - \frac{1}{3}BJ = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \neq 0$

No, not independent

c) $Y_1 + Y_2 + Y_3 = \vec{j}^T\vec{Y}$

$\vec{j}^T I A = \vec{j}^T (I - \frac{1}{n}J) = (0\ 0\ 0)$

Yes, independent.

5.32

a) $H = X(X^TX)^{-1}X^T$

$$HH = X(X^TX)^{-1}\underbrace{X^TX(X^TX)^{-1}}_{I}X^T$$

$$= X(X^TX)^{-1}X^T \quad \text{idempotent}$$

$(I-H)$ idempotent : $(I-H)(I-H) = I - H - H + HH$

$$= I - H - H + H$$

$$= I - H .$$

$\text{rank}(H) = \text{tr}(H)$

$= \text{tr}(X(X^TX)^{-1}X^T)$ $\qquad \text{tr}(AB) = \text{tr}(BA)$

$= \text{tr}((X^TX)(X^TX)^{-1}) = \text{tr}(I) = p .$

(X^TX) : $p\times n$ $n\times p$ $p\times p$

$\text{tr}(I-H) = n - p = \text{rank}(I-H).$

b) $\vec{\mu} = X\vec{b}$

$E(\vec{y}^TH\vec{y}) = \text{tr}(\sigma^2 HI) + \vec{\mu}^TA\vec{\mu}$

$$= \sigma^2 p + \vec{b}^TX^TX\underbrace{(X^TX)^{-1}X^TX}_{I}\vec{b}$$

$$= \sigma^2 p + \vec{b}^TX^TX\vec{b}$$

$$E(\vec{Y}^T(1-H)\vec{Y}) = tr(\sigma^2(I-H)) + \vec{b}^T X^T (I-H) X \vec{b}$$

$$= \sigma^2(n-p) + \vec{b}^T (X^TX - X^THX)\vec{b}$$

$$X^THX = X^TX(X^TX)^{-1}X^TX = X^TX$$

$$= \sigma^2(n-p)$$

c) $\frac{1}{\sigma^2}\vec{Y}^TH\vec{Y} \sim \chi^2(p,\lambda)$ $\quad \lambda = \frac{1}{2\sigma^2}\vec{b}^TX^TX\vec{b}$

$$\frac{1}{\sigma^2}\vec{Y}^T(I-H)\vec{Y} \sim \chi^2(n-p)$$

d) $H(1-H) = H - HH = H - H = 0 \Rightarrow$ independent

e) $$\frac{\frac{1}{\sigma^2}\vec{Y}^TH\vec{Y}/p}{\frac{1}{\sigma^2}\vec{Y}^T(1-H)\vec{Y}/(n-p)} \sim F(p, n-p, \lambda)$$