Assignment 5

Exam 2014

a) H symmetric, idempotent (projection matrix)

$H^T = H \qquad H \cdot H = H$

(lecture 7)

$\text{rank}(H) = p$

since $\text{rank}(H) = \text{trace}(H)$ (idempotent)

and $\text{trace}(AB) = \text{trace}(BA)$

$\text{trace}(H) = \text{tr}(I_p) = p$.

also L7.

HY: projection onto column space of X (L8)

b) We know that if $\vec{Y} \sim MVN(\vec{\mu}, \sigma^2 I)$, then $\frac{1}{\sigma^2}\vec{Y}^T A \vec{Y} \sim \chi^2(\text{rank}(A), \lambda)$ if A idempotent. $\lambda = \vec{\mu}^T A \vec{\mu} \frac{1}{2\sigma^2}$

corollary 2, thm 5.5 p 117-118

$(I-H)$ idempotent, $\text{rank}(I-H) = n-p$.

$\vec{\mu}^T A \vec{\mu} = (X\vec{\beta})^T (I-H) X\vec{\beta} = 0$ since $HX = X$.

$\frac{1}{\sigma^2} SSE \sim \chi^2(p)$.

$S^2 = \frac{SSE}{n-p}$ $\quad Var(S^2) = Var\left(\frac{\sigma^2}{n-p} \frac{SSE}{\sigma^2}\right)$

$= \frac{\sigma^4}{(n-p)^2} \underbrace{Var\left(\frac{SSE}{\sigma^2}\right)}$

$= 2(n-p)$ by χ^2-dist

$= \frac{2\sigma^4}{(n-p)}$

c) $A = (X^T X)^{-1} X^T$ $\quad B = I - H$

$p \times n \quad n \times p \quad p \times n$ $\quad n \times n$

$p \times p \quad p \times n$

$p \times n$

Indepence : check $A \Sigma B^T = A \sigma^2 I B$

$= \sigma^2 AB = 0$.

$AB = (X^T X)^{-1} X^T \cdot I - (X^T X)^{-1} \underbrace{X^T H}$

$= (X^T X)^{-1} X^T - (X^T X)^{-1} X^T$ $\quad (HX)^T = X^T$

$= 0$. ok.

$\hat{\beta} = A\vec{Y}$ $\quad SSE = \vec{Y}^T B \vec{Y} = (B\vec{Y})^T (B\vec{Y})$

Since $\vec{AY}$ and $\vec{BY}$ are independent,

$\vec{AY}$ and $(\vec{BY})^T(\vec{BY})$ are also independent

We e.g. use independence when we derive t-dist of

$$T = \frac{\hat{\beta}_j}{S\sqrt{g_{jj}}}$$

Exam 2024

a) use $t = \dfrac{\hat{\beta}_j}{\hat{SE}(\hat{\beta}_j)}$

$$\hat{SE}(\hat{\beta}_0) = \frac{-0.2119}{-0.518} = 0.409$$

$$t_1 = \frac{1.2285}{0.3170} = 3.875$$

$$\hat{\beta}_2 = 0.3170 \cdot 6.952 = 2.204$$

17 $$R^2_{adj} = \left(R^2 - \frac{k}{n-1}\right)\left(\frac{n-1}{n-k-1}\right)$$

$k = 2$

$n - k - 1 = n - 3 = 17 \qquad n = 20$

$$R^2_{adj} = \left(R^2 - \frac{2}{19}\right)\left(\frac{19}{17}\right) = 0.7208$$

b) $$\alpha_c = \frac{\alpha}{3} = \frac{0.01}{3} = 0.0033\ldots$$

reject (2) and (3).

c) 95% conf. int

Use $t_{0.025,17} = 2.110$

$\hat{\beta}_0 \pm 2.110 \cdot 0.409$

$[-1.07, 0.651]$

d) $n = 20$ $\quad \frac{3n}{4} = 15$ $\quad \frac{n}{4} = 5$

$$X = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{15} \quad \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}_{5} \quad \begin{matrix} \\ \vdots \\ 1 \end{matrix}_{15}$$

$$X^T X = \begin{bmatrix} 20 & 15 & 5 \\ 15 & 15 & 10 \\ 5 & 10 & 15 \end{bmatrix}$$

$(X^T X)^{-1}$ given

Method 1

$\widehat{Cov}(\hat{\beta}) = s^2 (X^T X)^{-1}$ $\quad Cov(\hat{\beta}_1, \hat{\beta}_2) = s^2 0.1$

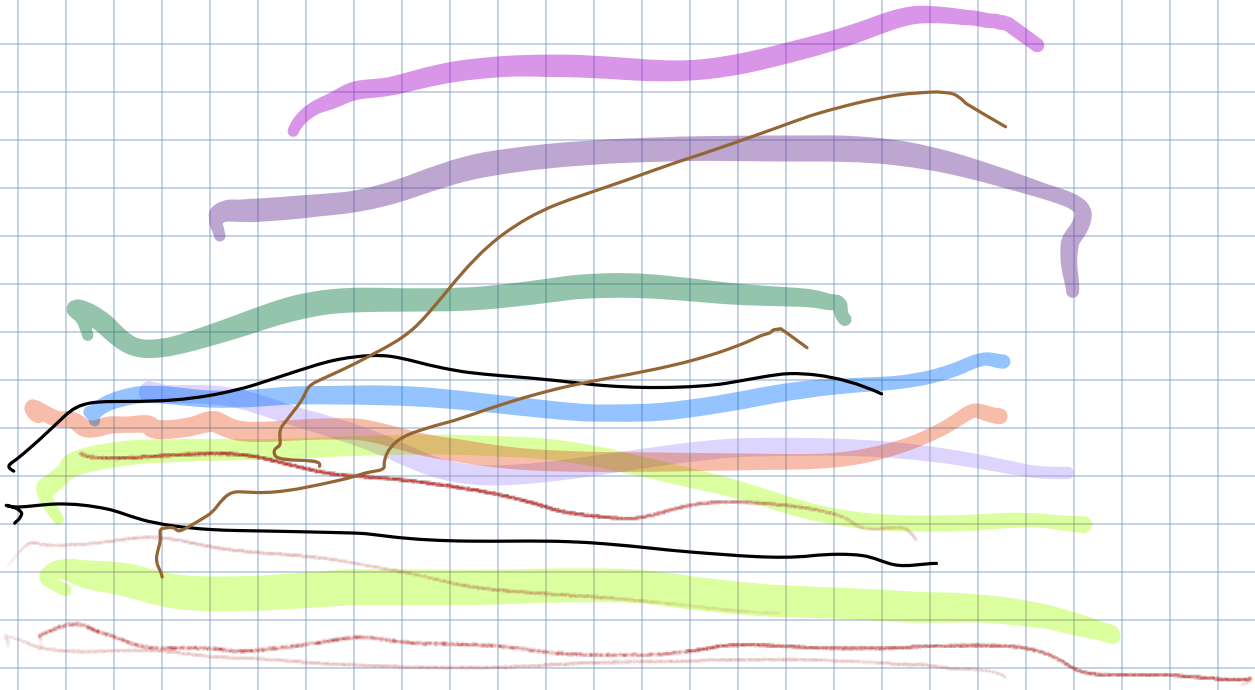
$$\hat{cor}(\hat{\beta}_1, \hat{\beta}_2) = \frac{\hat{cov}(\hat{\beta}_1, \hat{\beta}_2)}{\hat{SE}(\hat{\beta}_1)\,\hat{SE}(\hat{\beta}_2)} = \frac{s^2\, 0.1}{0.317^2}$$

$$s^2 = 0.5787^2$$

$$\approx 0.33$$

Method 2

$$cov(\hat{\beta}) = \frac{\sigma^2\, 0.1}{\sqrt{\sigma^2 0.3}\,\sqrt{\sigma^2 0.3}} = \frac{0.1}{0.3} = \frac{1}{3}$$



d) $\beta_1 = \beta_2 \rightarrow \beta_1 - \beta_2 = 0$

$C\vec{\beta} = \vec{0} \longrightarrow C = [0 \;\; 1 \;\; -1]$

$$F = \frac{(C\hat{\beta})^T (C(X^TX)^{-1}C^T)^{-1} C\hat{\beta}}{SSE/n-k-1}$$

49

$$C\hat{\beta} = \hat{\beta}_1 - \hat{\beta}_2 = 1.0285 - 2.004 = -0.9755$$

$$(X^TX)^{-1} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.2 \\ 0.2 \end{bmatrix}$$

$[0 \;\; -1 \;\; 1] \rightarrow = 0.4$

$$F = \frac{(-0.9755)^2}{0.4 \cdot 0.5787^2} = 7.104$$

$F_{0.05,1,17} = 4.45 \rightarrow$ reject

