

TMA4267 exam spring 2026: solution manual and grading document

At most 10 points per sub-problem, and grade set by 'prosentvurderingsmetoden', <https://i.ntnu.no/wiki/-/wiki/norsk/prosentvurderingsmetoden> but also with regard to the general NTNU grade descriptions. <https://i.ntnu.no/wiki/-/wiki/Norsk/Karakterskalaen> Suggestion for points given in red below, but the entire sub-problem also always considered as a whole. Other correct solution methods also awarded full marks.

Problem 1

- (a) Since $\mathbf{Y}$ is multivariate normal, $Y_2 \sim N(\mu_2, \Sigma_{22})$. From the given mean vector and covariance matrix,

$$\mu_2 = -1, \quad \Sigma_{22} = 2.$$

Hence $Y_2 \sim N(-1, 2)$. [5 points]

For a multivariate normal distribution,

$$E(Y_2 \mid Y_1 = y_1) = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(y_1 - \mu_1).$$

Using $\mu_1 = 1$, $\mu_2 = -1$, $\Sigma_{21} = 1$, $\Sigma_{11} = 1$, we get

$$E(Y_2 \mid Y_1 = y_1) = -1 + 1(y_1 - 1) = y_1 - 2.$$

Thus the expression can be written as $\beta_0 + \beta_1 y_1$, with $\beta_0 = -2$, $\beta_1 = 1$. [5 points]

- (b) Define

$$\mathbf{Z} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -1 \end{bmatrix} \mathbf{Y} = \mathbf{A}\mathbf{Y}.$$

Since linear transformations of multivariate normal vectors are multivariate normal, $\mathbf{Z} \sim \text{MVN}_2(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\Sigma\mathbf{A}^\top)$. [2 points]

The mean vector becomes $E(\mathbf{Z}) = \mathbf{A}\boldsymbol{\mu} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. [3 points]

The covariance matrix becomes $\text{Cov}(\mathbf{Z}) = \mathbf{A}\Sigma\mathbf{A}^\top = \begin{bmatrix} 10 & 6 \\ 6 & 18 \end{bmatrix}$. [5 points]

Problem 2

- (a) There were 2 levels for each factor since both have $\text{Df} = 1$. [2 points]

Since the total number of observations is $n = 36 + 3 + 1 = 40$, and there are 4 factor level combinations, there were 10 experiments at each combination of factor levels. [3 points]

The missing values are found as follows:

$$\text{Df}_{A:B} = (2 - 1)(2 - 1) = 1,$$

$$\text{MS}_{\text{Residuals}} = \frac{704.64}{36} = 19.5733,$$

and

$$F_{A:B} = \frac{23.46}{19.5733} = 1.1986.$$

[1 ? = 2 points, 2 ? = 3 points, 3 ? = 5 points]

Problem 3

(a) We compute

$$H^2 = X(X^\top X)^{-1}X^\top X(X^\top X)^{-1}X^\top.$$

Since $X^\top X(X^\top X)^{-1} = I$, we get $H^2 = X(X^\top X)^{-1}X^\top = H$, so H is idempotent. [5 points]

Using that rank equals trace for idempotent matrices,

$$\text{rank}(H) = \text{tr}(H) = \text{tr}\left(X(X^\top X)^{-1}X^\top\right).$$

Using cyclic trace property for square matrices $\text{tr}(AB) = \text{tr}(BA)$,

$$\text{tr}(H) = \text{tr}\left(X^\top X(X^\top X)^{-1}\right) = \text{tr}(I_p) = p.$$

[5 points]

(b) Using the formula

$$E(\mathbf{Y}^\top \mathbf{A} \mathbf{Y}) = \text{tr}(\mathbf{A} \Sigma) + \boldsymbol{\mu}^\top \mathbf{A} \boldsymbol{\mu},$$

with $\boldsymbol{\mu} = X\boldsymbol{\beta}$, $\Sigma = \sigma^2 I$, we obtain $E(\mathbf{Y}^\top H \mathbf{Y}) = \text{tr}(H\sigma^2 I) + (X\boldsymbol{\beta})^\top H(X\boldsymbol{\beta})$. Since $\text{tr}(H) = p$ and $HX = X$,

$$E(\mathbf{Y}^\top H \mathbf{Y}) = \sigma^2 p + \boldsymbol{\beta}^\top X^\top X \boldsymbol{\beta}.$$

[5 points]

Furthermore,

$$\frac{1}{\sigma^2} \mathbf{Y}^\top H \mathbf{Y} \sim \chi^2(r, \lambda),$$

where $r = \text{rank}(H) = p$, and $\lambda = \frac{1}{2\sigma^2} (X\boldsymbol{\beta})^\top H(X\boldsymbol{\beta}) = \frac{1}{2\sigma^2} \boldsymbol{\beta}^\top X^\top X \boldsymbol{\beta}$. [5 points]

(c) Define $U = \frac{1}{\sigma^2} \mathbf{Y}^\top \mathbf{H} \mathbf{Y} \sim \chi^2(p, \lambda)$, a non-central chi-squared distribution.

Also, define

$$V = \frac{1}{\sigma^2} \mathbf{Y}^\top (\mathbf{I} - \mathbf{H}) \mathbf{Y},$$

which must have a chi-squared distribution ($\mathbf{I} - \mathbf{H}$ is idempotent) with $\text{rank}(\mathbf{I} - \mathbf{H}) = (n - p)$ dof. Since $(\mathbf{I} - \mathbf{H})\mathbf{X} = 0$, the non-centrality parameter becomes

$$\lambda_2 = \frac{1}{2\sigma^2} (\mathbf{X}\boldsymbol{\beta})^\top (\mathbf{I} - \mathbf{H})(\mathbf{X}\boldsymbol{\beta}) = 0.$$

Therefore, $V \sim \chi^2(n - p)$, a central chi-square distribution. [5 points]

Moreover,

$$\mathbf{H}\sigma^2\mathbf{I}(\mathbf{I} - \mathbf{H}) = \sigma^2(\mathbf{H} - \mathbf{H}^2) = \sigma^2(\mathbf{H} - \mathbf{H}) = 0,$$

so U and V are independent. [5 points] Hence

$$Q = \frac{U/p}{V/(n - p)} \sim F(p, n - p, \lambda).$$

Problem 4

$$\mathbb{E}(\tilde{\boldsymbol{\beta}}_1) = (\mathbf{X}_1^\top \mathbf{X}_1)^{-1} \mathbf{X}_1^\top \mathbb{E}(\mathbf{Y}) = (\mathbf{X}_1^\top \mathbf{X}_1)^{-1} \mathbf{X}_1^\top (\mathbf{X}_1 \boldsymbol{\beta}_1 + \mathbf{X}_2 \boldsymbol{\beta}_2).$$

Hence $\mathbb{E}(\tilde{\boldsymbol{\beta}}_1) = \boldsymbol{\beta}_1 + (\mathbf{X}_1^\top \mathbf{X}_1)^{-1} \mathbf{X}_1^\top \mathbf{X}_2 \boldsymbol{\beta}_2$.

Thus $\tilde{\boldsymbol{\beta}}_1$ is unbiased for $\boldsymbol{\beta}_1$ if

$$\mathbf{X}_1^\top \mathbf{X}_2 = 0,$$

that is, if $\mathbf{X}_2$ and $\mathbf{X}_1$ have orthogonal columns. [5 points] No requirements for $\mathbf{V}$ [5 points, also given if not explicitly stated].

Problem 5

(a) A 95% confidence interval becomes

$$78.02 \pm 2.080 \cdot 56.72,$$

which gives

$$[-39.96, 195.99] \approx [-40, 196].$$

[5 points]

The estimated coefficient of premium location is 78.02, representing an expected turnover for a premium location of approximately 78 000 NOK higher than for a

non-premium location, with other variables held constant. However, this estimate has large uncertainty, and a test of

$$H_0 : \beta_3 = 0 \quad \text{against} \quad H_1 : \beta_3 \neq 0$$

cannot be rejected at the 5% significance level, corresponding to the confidence interval including 0. [5 points if the discussion of the effect size shows some knowledge of the estimate being positive and consideration of the wide interval or high p-value]

- (b) From the QQ-plot, the normality assumption appears reasonable. From the plot of fitted values against residuals, there may be some indication of non-constant variance, but with few observations it is difficult to conclude. [5 points, if the discussion is not completely generic and needs to show uncertainty in conclusion due to few observations]

The residual vector is $\hat{\varepsilon} = (I - H)\mathbf{Y}$, while the fitted values are $\hat{\mathbf{Y}} = H\mathbf{Y}$. Hence

$$\text{Cov}(\hat{\varepsilon}, \hat{\mathbf{Y}}) = (I - H)\sigma^2 I H = \sigma^2(I - H)H = 0.$$

[5 points]

- (c) With repeated measurements of the same store, the assumption of independent observations (uncorrelated errors) is likely violated. [5 points]

For repeated measurements from the same store Y_{ij} and $Y_{ij'}$, the correlation between them is

$$\text{Corr}(Y_{ij}, Y_{ij'}) = \frac{\tau^2}{\tau^2 + \sigma^2}.$$

The estimated correlation is approximately 0.72, indicating relatively strong dependence between observations from the same store and substantial variation between stores. [5 points if the answer is valid and connects the number to the actual problem]