

Two-way ANOVA (cont.)

Model:

$$Y_{ijk} = \mu + \alpha_i + \gamma_j + (\alpha\gamma)_{ij} + \epsilon_{ijk}$$

Annotations:

- α_i : effect factor α , level i
- μ : global mean
- γ_j : effect factor γ , level j
- $(\alpha\gamma)_{ij}$: added effect of combining levels i, j of two factors
- ϵ_{ijk} : $N(0, \sigma^2)$ noise
- Y_{ijk} : response at level i , level j , "subject" k

$$\sum_i \alpha_i = 0 \quad \sum_j \gamma_j = 0 \quad \sum_i (\alpha\gamma)_{ij} = 0 \quad \sum_j (\alpha\gamma)_{ij} = 0$$

[Slides formula sheet + age/memory]

Example:

Y : words recalled (of 27)

α : age young / old $m_\alpha = 2$

γ : process count, rhyme, ... $m_\gamma = 5$

Aim $H_0: (\alpha\gamma)_{ij} = 0$
for all ij

$H_1: (\alpha\gamma)_{ij} \neq 0$
at least one ij .

Tool: LM + partial F-test

$$\vec{Y} = X \vec{\beta} + \vec{\varepsilon}$$

$$\vec{\varepsilon} \sim \text{MVN}(0, \sigma^2 I)$$

$n=100$
random
responses

$$\vec{\beta} = \begin{bmatrix} \mu \\ \alpha_1 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix}$$

$$\alpha_2 = -\alpha_1$$

$$\gamma_5 = -\sum_{j=1}^4 \gamma_j$$

$$\vec{\beta}_{\alpha\gamma} = \begin{bmatrix} (\alpha\gamma)_{11} \\ (\alpha\gamma)_{12} \\ (\alpha\gamma)_{13} \\ (\alpha\gamma)_{14} \end{bmatrix}$$

$$(\alpha\gamma)_{21} = -(\alpha\gamma)_{11}$$

$$(\alpha\gamma)_{22} = -(\alpha\gamma)_{12}$$

$\vdots$

$$(\alpha\gamma)_{15} = -\sum_j (\alpha\gamma)_{1j}$$

$X^?$ write row of X for young and counting

level 2 α

level 1 γ

$$\vec{X}^T = [1 \ -1 \ 1 \ 0 \ 0 \ 0 \ -1 \ 0 \ 0 \ 0]$$

$$E(Y | \text{young, count}) = \vec{X}^T \vec{\beta}$$

$$= \mu - \alpha_1 + \gamma_1 - (\alpha\gamma)_{11}$$

$$= \mu + \alpha_2 + \gamma_1 + (\alpha\gamma)_{21}$$

View
slide

ANOVA Table

$$(1) : 4 \quad (m_x - 1)(m_y - 1) = 1 \cdot 4 = 4.$$

$$\text{Sum sq: } \underset{\substack{\uparrow \\ \text{full model}}}{SSR} - \underset{\substack{\uparrow \\ \text{model with no interactions}}}{SSR^*} = 190.30$$

$$F = \frac{(SSR - SSR^*) / 4 \leftarrow (m_x - 1)(m_y - 1)}{SSE / 90 \leftarrow n - k - 1} = \frac{47.58}{8.03} = 5.927.$$

$$p = 0.00027..$$

highly unlikely to see our data (slide)
if true.

NB (if time) : conclusion relies on model assumptions

$$E_{ijk} \sim N(0, \sigma^2)$$

here: normal approximation

since y discrete $\{0, 1, \dots, 27\}$
and bounded

constant variance?

QQ plot ok.

If time

Young, imagery (process 4)

$$\hat{\mu}_{24} = 11.61 + 1.55 + 3.89 + 0.55 = 17.6$$

young effect imagery effect added combination effect (minimal) words recalled

Old, counting (process 1)

$$\hat{\mu}_{11} = 11.61 - 1.55 - 4.86 + \boxed{1.6?}$$

counting effect old effect ?

should reverse old effect, there is no difference between ages.

DOE : 2-level factorial designs

Example 1 :

$k=3$ factors 2^3 factorial design
each at two levels

All 8 factor level combinations
measured twice.

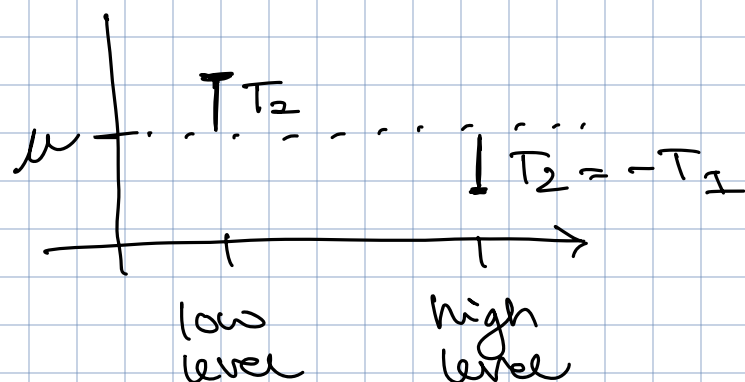
Model 1 : without interactions (ANOVA)

$$Y_{ijkl} = \mu + T_i + C_j + P_{T_k} + \varepsilon_{ijkl}$$

$$i=1,2 \quad j=1,2 \quad k=1,2$$

$l=1,2$
(2 replicates
for each
 ijk .)

$T_1, T_2 :$



T_2 : Deviation from
global mean due
to level 2, factor T.

$2T_2$: effect of
changing from
low to high level
of factor T.

most
interesting
here.

Convention
to have low level
as -1 .

← slightly
different from our
more complex ANOVA models.

Tool: $\vec{y} = X\vec{\beta} + \vec{\varepsilon}$

Design matrix X , one run (2^3)

$$X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

x_1	x_2	x_3
T	C	Ph

$$\vec{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

μ
 effect of
 changing
 x_1 one unit

↓

$2\beta_1$

change
 x_1 from
 low to
 high
 level