

TMA4267 Linear statistical models

Spring 2026

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From formula sheet

ANOVA

One-way:

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}, \quad i = 1, \dots, m, \quad j = 1, \dots, n_i, \quad \sum_i \alpha_i = 0.$$

Two-way:

$$Y_{ijk} = \mu + \alpha_i + \gamma_j + (\alpha\gamma)_{ij} + \varepsilon_{ijk}, \quad i = 1, \dots, m_\alpha, \quad j = 1, \dots, m_\gamma, \quad k = 1, \dots, n_{ij},$$

$$\sum_i \alpha_i = 0, \quad \sum_j \gamma_j = 0, \quad \sum_i (\alpha\gamma)_{ij} = 0, \quad \sum_j (\alpha\gamma)_{ij} = 0.$$

Two-way ANOVA with interactions

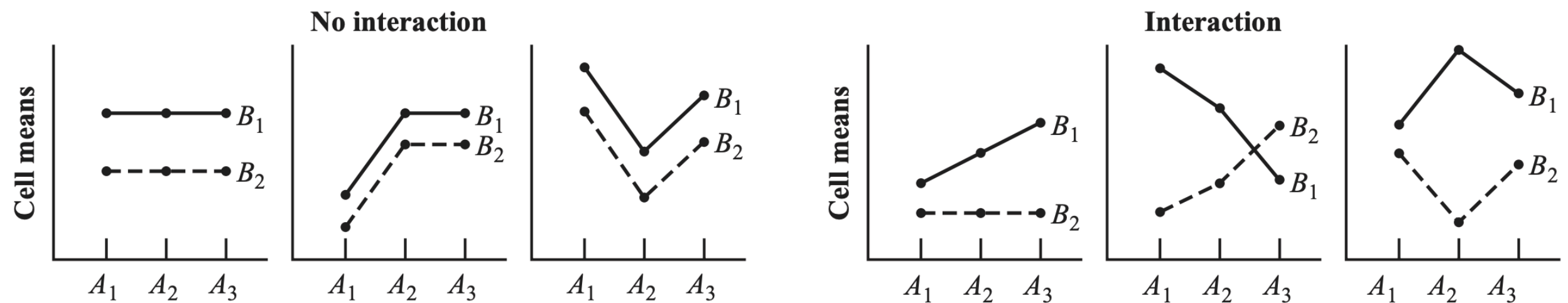


Figure 13.2 Illustration of possible noninteractions and interactions

From the textbook *Statistical methods for psychology* by Howell (2010)

Two-factor anova

Age and memory data

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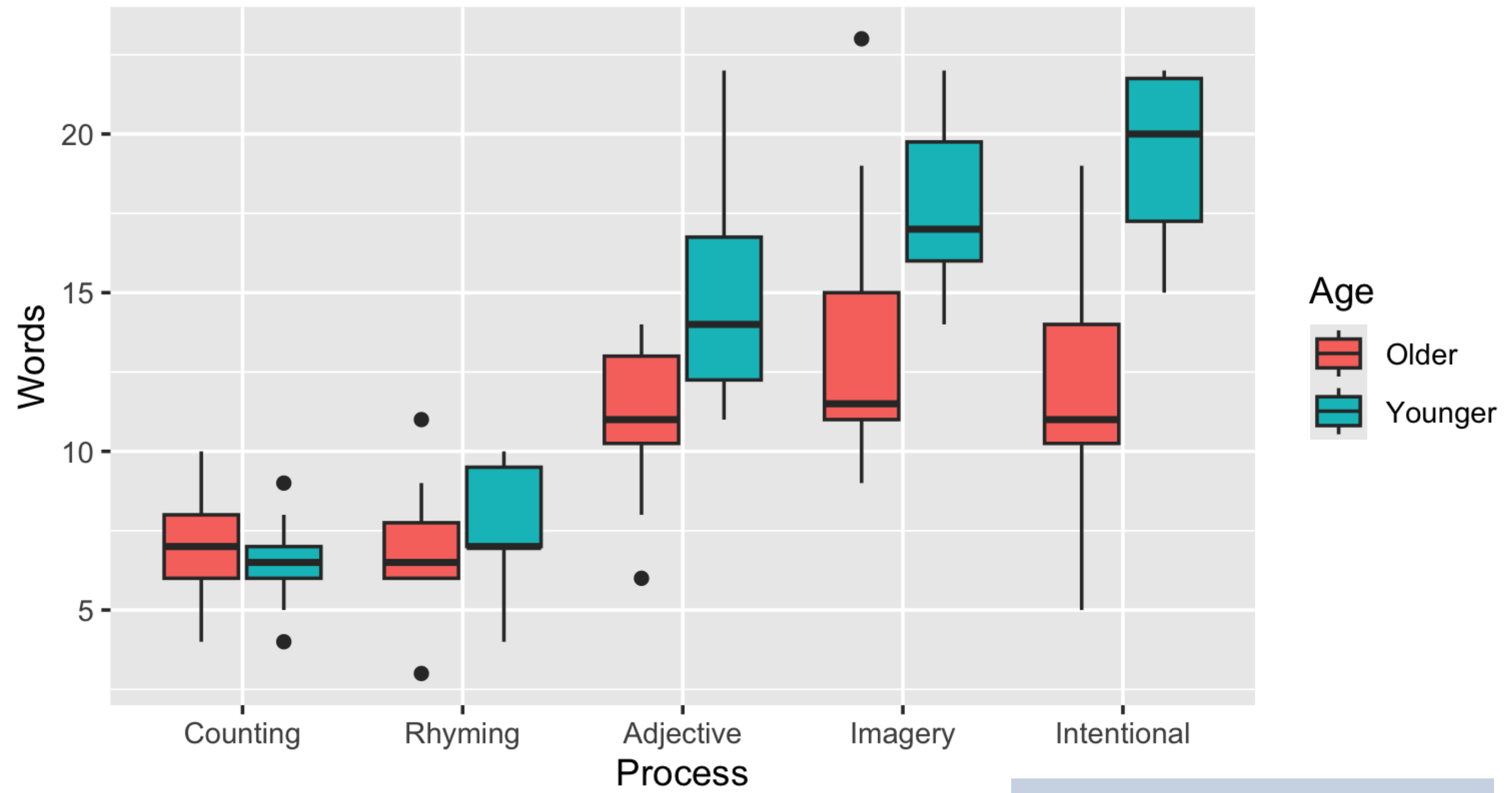
11.1 An Example

Many features of the analysis of variance can be best illustrated by a simple example, so we will begin with a study by M. W. Eysenck (1974) on recall of verbal material as a function of the level of processing. The data we will use have the same group means and standard deviations as those reported by Eysenck, but the individual observations are fictional.

Two-factor anova

Age and memory data

- Eysenck study (1974)
 - 50 older (55-65 years) participants randomly split into 5 groups (10 per group)
 - 50 younger (18-30 years) participants randomly split into 5 groups (10 per group)
 - All groups: read list of 27 words
 - Group 1: Count letters in each word
 - Group 2: Find words that rhyme
 - Group 3: Add adjectives to each word
 - Group 4: Imagery
 - Group 5: Intentionally aim to recall words
 - Afterwards: how many words from the list could the participants recall?



"By chance" or evidence of an interaction effect?

```
> head(ds,10)
```

	Age	Process	Words
1	Younger	Counting	8
2	Younger	Counting	6
3	Younger	Counting	4
4	Younger	Counting	6
5	Younger	Counting	7
6	Younger	Counting	6
7	Younger	Counting	5
8	Younger	Counting	7
9	Younger	Counting	9
10	Younger	Counting	7

|

```
> model.matrix(model)
```

	(Intercept)	Age1	Process1	Process2	Process3	Process4	Age1:Process1	Age1:Process2	Age1:Process3	Age1:Process4
1	1	-1	1	0	0	0	-1	0	0	0
2	1	-1	1	0	0	0	-1	0	0	0
3	1	-1	1	0	0	0	-1	0	0	0
4	1	-1	1	0	0	0	-1	0	0	0
5	1	-1	1	0	0	0	-1	0	0	0
6	1	-1	1	0	0	0	-1	0	0	0
7	1	-1	1	0	0	0	-1	0	0	0
8	1	-1	1	0	0	0	-1	0	0	0
9	1	-1	1	0	0	0	-1	0	0	0
10	1	-1	1	0	0	0	-1	0	0	0
11	1	-1	0	1	0	0	0	-1	0	0
12	1	-1	0	1	0	0	0	-1	0	0
13	1	-1	0	1	0	0	0	-1	0	0
14	1	-1	0	1	0	0	0	-1	0	0
15	1	-1	0	1	0	0	0	-1	0	0

```
> anova(model)
```

Analysis of Variance Table

Response: Words

	Df	Sum Sq	Mean Sq	F value	Pr(>F)	
Age	1	240.25	240.25	29.9356	3.981e-07	***
Process	4	1514.94	378.74	47.1911	< 2.2e-16	***
Age:Process	(1)	190.30	47.58	(2)	0.0002793	***
Residuals	90	722.30	8.03			

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

```
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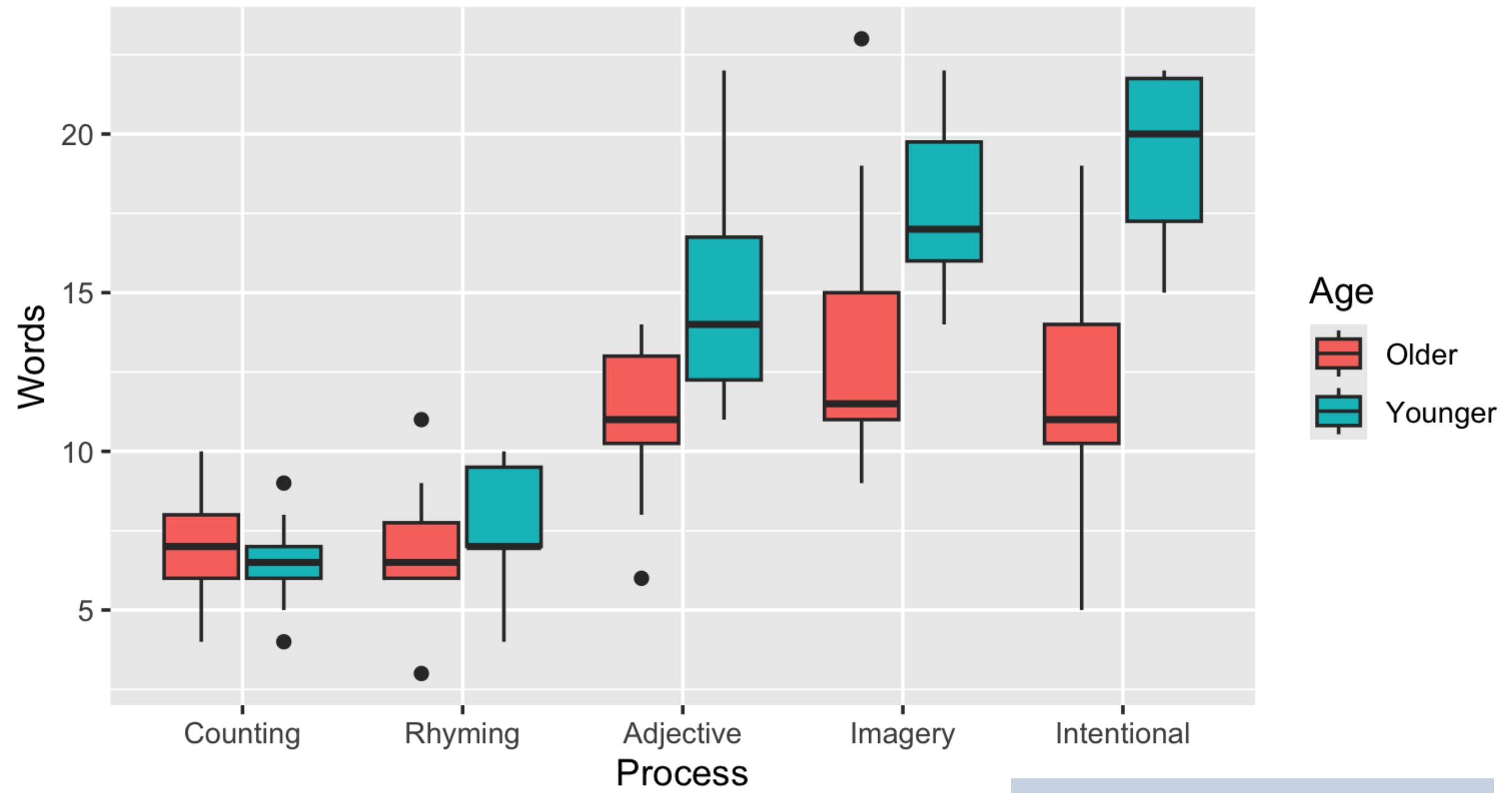
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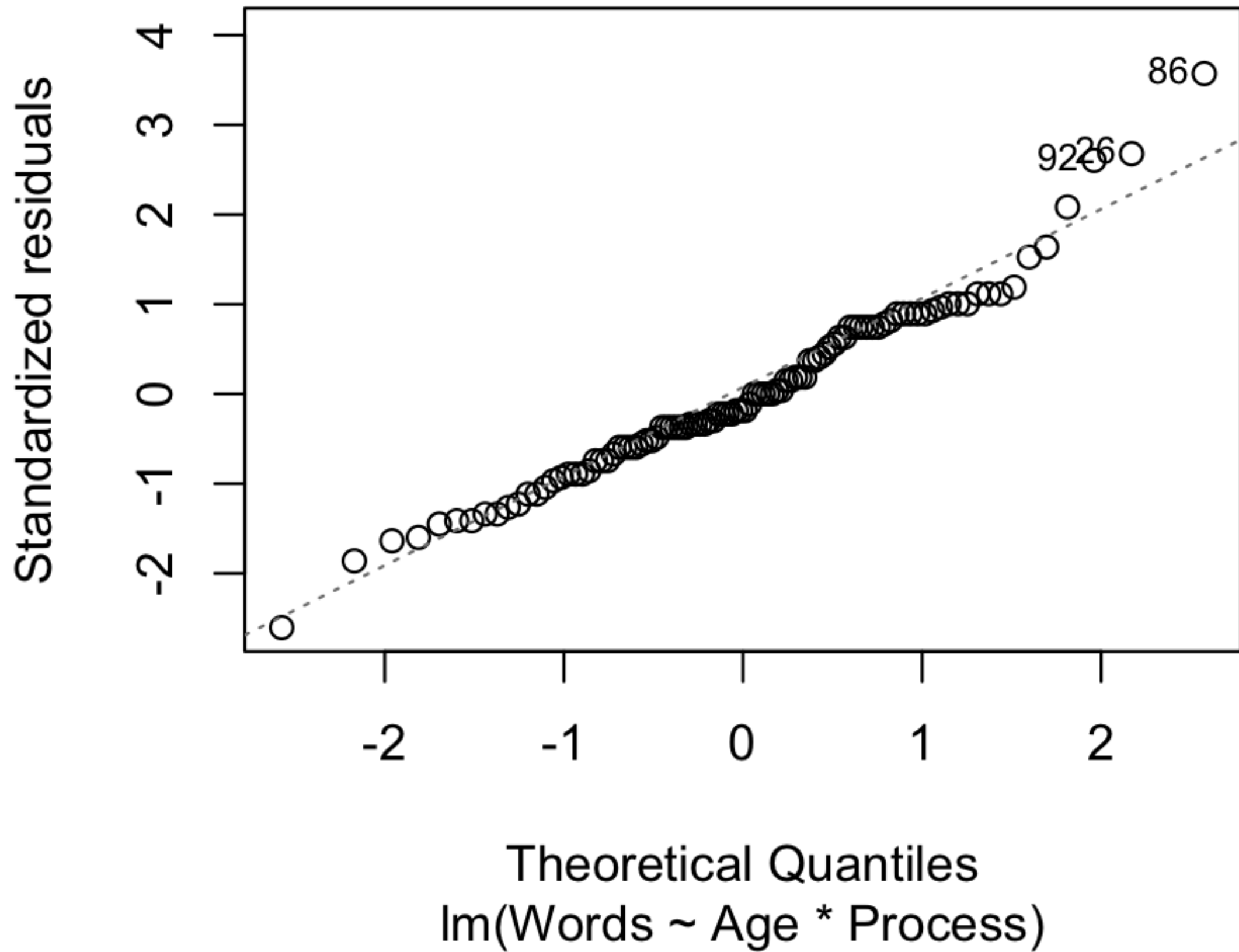
```
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```

```
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```



"By chance" or evidence of an interaction effect?

Q-Q Residuals



Call:

```
lm(formula = Words ~ Age * Process, data = ds)
```

Residuals:

Min	1Q	Median	3Q	Max
-7.0	-1.6	-0.5	2.0	9.6

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)	
(Intercept)	11.6100	0.2833	40.982	< 2e-16	***
Age1	-1.5500	0.2833	-5.471	3.98e-07	***
Process1	-4.8600	0.5666	-8.578	2.60e-13	***
Process2	-4.3600	0.5666	-7.695	1.72e-11	***
Process3	1.2900	0.5666	2.277	0.02517	*
Process4	3.8900	0.5666	6.866	8.24e-10	***
Age1:Process1	1.8000	0.5666	3.177	0.00204	**
Age1:Process2	1.2000	0.5666	2.118	0.03694	*
Age1:Process3	-0.3500	0.5666	-0.618	0.53831	
Age1:Process4	-0.5500	0.5666	-0.971	0.33429	

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2.833 on 90 degrees of freedom

Multiple R-squared: 0.7293, Adjusted R-squared: 0.7022

F-statistic: 26.93 on 9 and 90 DF, p-value: < 2.2e-16

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Coefficients:

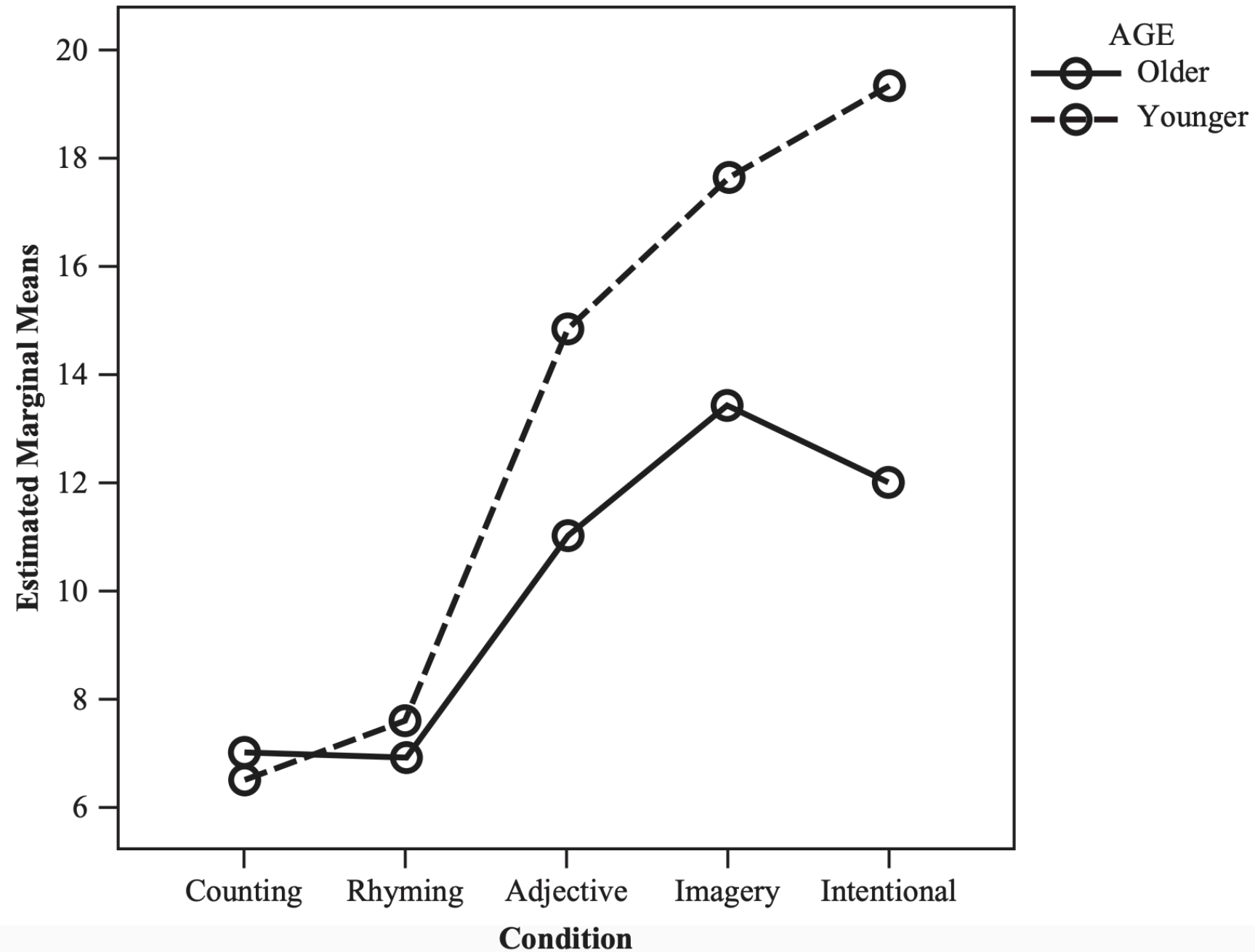
	Estimate	Std.
(Intercept)	11.6100	
Age1	-1.5500	
Process1	-4.8600	
Process2	-4.3600	
Process3	1.2900	
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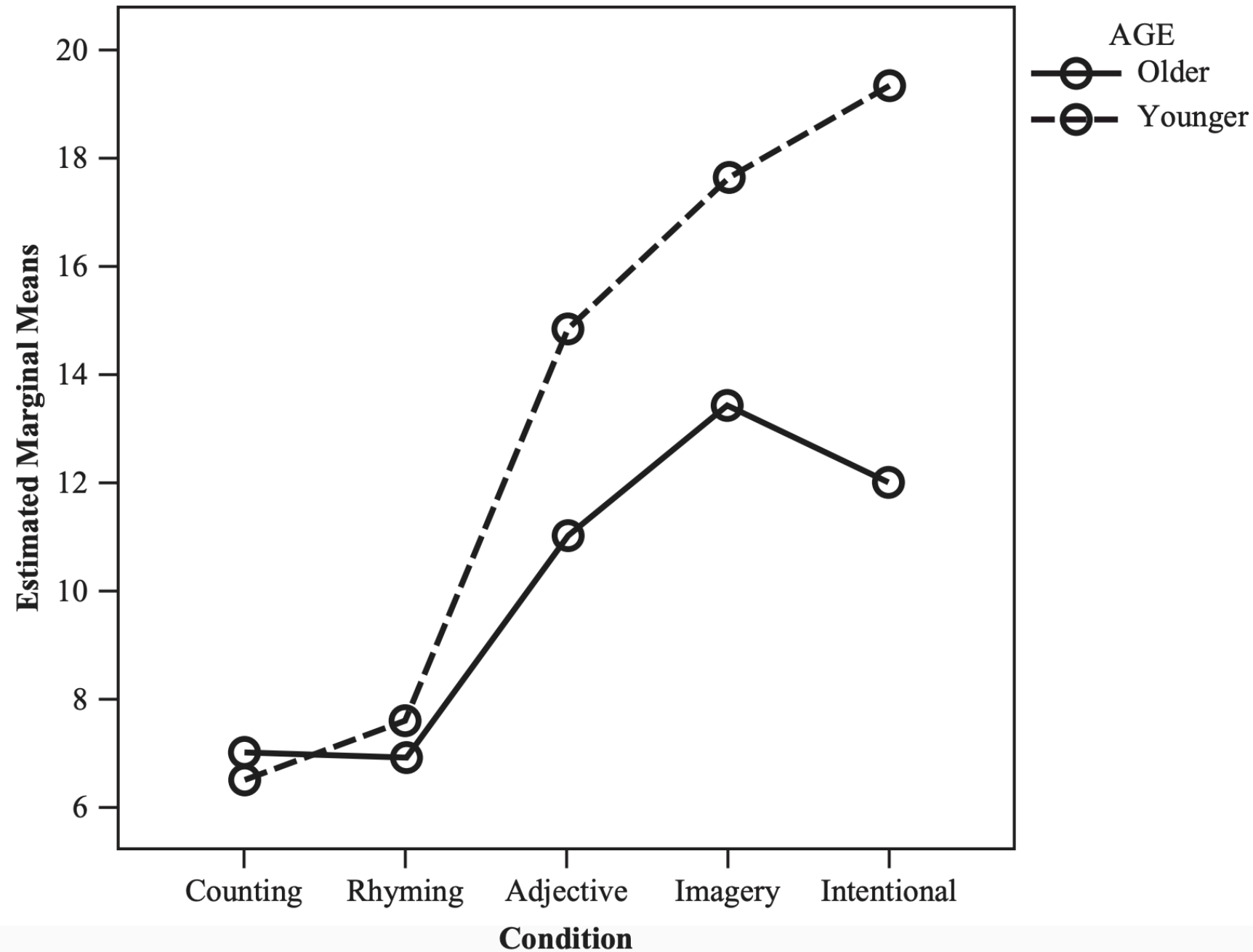
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Design of experiments (DOE) and 2-level factorial designs

First: What is "wrong" with observational studies?

Example 1:

In Norway, the rate of occurrence of newborn babies with severe conditions is greater in large hospitals than in small.

Are smaller hospitals safer?

Example 2:

Children who live in homes with many books do better in school.

Should all homes have more books?

Design of experiments

- If I change (vary) input x so and so, what happens to my response y ?
 - How can I make sure that changes in y occurred due to the varying of x and not other causes?
 - How can I get as much understanding as possible from few (expensive) experiments?
-
- ➡ Choose appropriate levels for x
 - ➡ Randomise experiments and avoid confounding
 - ➡ Do multifactorial experiments (not one-factor-at-a-time)
-
- ★ Analysis tool: ANOVA and linear models

2 level factorial designs

- Aim: What affects our response Y , and by how much?
- Procedure:
 - Select k factors
 - Select **two** levels for each factor
 - "High" and "low" level
 - One experimental run: measure Y at all 2^k factor level combinations
 - Repeat if necessary

Example 1



Process Biochemistry 40 (2005) 779–788

PROCESS
BIOCHEMISTRY

www.elsevier.com/locate/procbio

Biosorption of chromium using factorial experimental design

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Received 10 October 2003; accepted 5 February 2004

<https://doi.org/10.1016/j.procbio.2004.02.024>

2^3 full factorial design, duplicated

Wikipedia:

Biosorption can be defined as the ability of biological materials to accumulate heavy metals from wastewater through metabolically mediated or physico-chemical pathways of uptake.^[2] Though using biomass in environmental cleanup has been in practice for a while, scientists and engineers are hoping this phenomenon will provide an economical alternative for removing toxic heavy metals from [industrial wastewater](#) and aid in [environmental remediation](#).

Example 1: Biosorption of chromium using factorial experimental design

- Aim: Removal of Cr^{3+} (and Cr^{6+})
- Design: 2^3 factorial design
- Factors:
 - T : temperature (29 and 55 degrees celcius)
 - C : metal concentration (10 and 1200 mg/L)
 - pH : pH (2.0 and 6.0)
- Response: Efficiency of chromium removal after 6h exposition time