

① Basic theory

random vectors $(\vec{\mu}, \Sigma)$

$$H_{\text{MN}}(\vec{\mu}, \Sigma)$$

↓

linear

↓
-1
0
↓

Quadratic

independence

 χ^2, F, t

② Model inference

$$p = x_1 x_2 \dots x_n$$

$$\hat{\beta} \sim \text{MWN}(\hat{\beta}, \sigma^2 (X^T X)^{-1})$$

[illegible]

$$s^2 = \text{SSE} / (n-k-1)$$

$$= \vec{y}^T (I - H) \vec{y} / (n-k-1)$$

$$SSE/\sigma^2 \sim \chi^2_{(n-k-1)}$$

β , SE independent

$$\begin{array}{c} \downarrow \\ \text{K} \\ \text{||} \\ \text{X} \\ \downarrow \\ \text{O} \\ + \\ \text{O} \\ \downarrow \end{array}$$

$$\Rightarrow F \text{ and } t$$

e.g.

$$H_0: \beta_j = 0$$

$$SE(\hat{\beta}_j) + t_{\alpha/2}$$

more advanced

$H_0: \beta = 0 \rightarrow t$ tests

(small/ restricted model
vs large model)

→ Generalize
→ Specialize
→ Total
→ our self

③ Model application

Interpret model results,
use model summary,
make predictions,

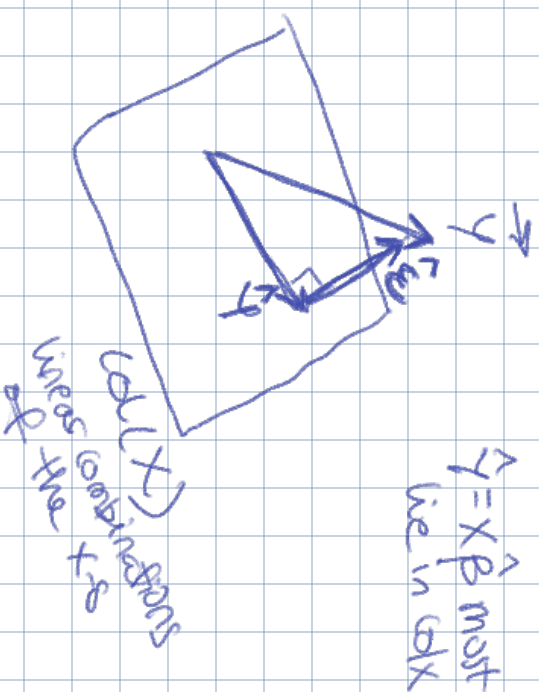
situation $\rightarrow$ model $\rightarrow$ situation

Check model:

R^2 , overall $\bar{r}$
Residual plots.

e.g. $\mathbb{M}_2$ vs Δ .
 $\mathbb{M}_2 = \mathbb{R}^2$

$2+3$ * Geometrically



extensions

④ OLS assumptions
too strict

$$\vec{Y} = X \vec{\beta} + \vec{\varepsilon}$$

GLS: $\vec{\varepsilon} \sim \text{MVN}(\vec{0}, \sigma^2 V)$, known V

e.g. heterosk., correlations

$$\vec{\beta} = (X^T V^{-1} X)^{-1} X^T V^{-1} \vec{Y}$$

$$\text{LMU: } \vec{y} = X \vec{\beta} + U \gamma + \vec{\varepsilon}$$

fixed effects $\rightarrow$ random effects $\rightarrow$

$$\vec{\gamma} \sim \text{MVN}(\vec{0}, Q) \quad Q \text{ unknown}$$

$$\vec{\varepsilon} \sim \text{MVN}(\vec{0}, \sigma^2 I) \quad \leftrightarrow \text{independent}$$

$$\text{Cov}(\vec{y}) = U Q U^T + \sigma^2 I \quad " = \sigma^2 V "$$

(estimate $Q \rightarrow \text{GLS}$)

Comparing groups/
factor levels.

⑤ Special cases

ANOVA (factors)

$$Y = \mu + \alpha + \gamma + (\alpha\gamma) + \varepsilon$$

(effects model)

Anova table / partial F-test

Finding important
factors.

Factorial designs

2 levels $-1, +1$

Effect estimates
& interactions.

NB all these
parts are also
'stand-alone'

Exercise 1: Prove $\hat{\beta}$ and SSE independent

$$\vec{y} \sim \text{MVN}(\vec{\mu}, \Sigma) \quad \vec{\mu} = X\vec{\beta}, \quad \Sigma = \sigma^2 I$$

$$\hat{\beta} = \underbrace{(X^T X)^{-1} X^T}_{B} \vec{y} = \underbrace{B}_{(k+1) \times n} \underbrace{\vec{y}}_{n \times 1}$$

$$\text{SSE} = \vec{y}^T \underbrace{(I - H)}_A \vec{y}$$

A $n \times n$ symm.

NB updated
formula
sheet

Formula sheet: $B \Sigma A = 0$?

$$\begin{aligned} & (X^T X)^{-1} X^T \sigma^2 I (I - H) \\ &= \sigma^2 (X^T X)^{-1} \underbrace{X^T (I - H)} \end{aligned}$$

$$X^T (I - H) = [(I - H) X]^T$$

$$IX = X$$

$$HX = X \quad \text{why?}$$

$$X (X^T X)^{-1} X^T X$$

but easier (?) to
think of H as projection
onto column space of
 X . Therefore HX
does nothing to X .

Exercise 2:

We measure output of several machines

over time : $i = 1, \dots, n$ machines

$j = 1, \dots, m_i$: repeated measurements per mach.

LM: $Y_{ij} = \beta_0 + \beta_1 t_{ij} + \gamma_i + \epsilon_{ij}$
 $\nwarrow$ time trend

$$\delta_i \sim N(0, \tau^2)$$

$$\epsilon_{ij} \sim N(0, \sigma^2)$$

> independent

$$\text{Var}(Y_{ij}) = \text{Var}(\gamma_i) + \text{Var}(\varepsilon_{ij}) = \tau^2 + \sigma^2$$

$$\begin{aligned}\text{Cov}(Y_{ij}, Y_{ij'}) &= \text{Cov}(\gamma_i + \varepsilon_{ij}, \gamma_i + \varepsilon_{ij'}) \\ &= \text{Cov}(\gamma_i, \gamma_i) = \text{Var}(\gamma_i) = \tau^2\end{aligned}$$

$$\text{corr}(y_{ij}, y_{ij'}) = \frac{\tau^2}{\tau^2 + \sigma^2}$$

"intra class correlation"

Machine :

↓
offset γ_i

correlated observations

Result : $\hat{T}^2 = 4$ $\hat{\sigma}^2 = 1$

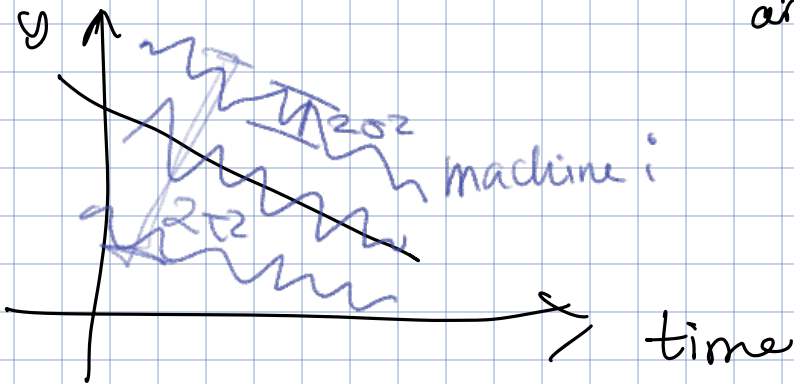
- Interpret

$\hat{\tau}^2 = 4$ between-machine variability

$\hat{\sigma}^2 = 1$ within-machine variability

$$\frac{\hat{\tau}^2}{\hat{\tau}^2 + \hat{\sigma}^2} = \frac{4}{5} = 0.8 \quad \text{high correlation}$$

repeated measurements
are similar.



Random slope

$$y_{ij} = \beta_0 + \beta_1 t_{ij} + \gamma_{0i} + \gamma_{1i} t_{ij} + \varepsilon_{ij}$$

$$\vec{\gamma}_i = \begin{pmatrix} \gamma_{0i} \\ \gamma_{1i} \end{pmatrix} \sim \text{MVN}(\mathbf{0}, \mathbf{\Sigma}) \quad \varepsilon_{ij} \sim \text{N}(0, \sigma^2)$$

