

Note : Definition of multivariate normal

• $\vec{Y} \sim \text{MNN}_n(\vec{\mu}, \Sigma)$ if

$$f(\vec{y}) = \frac{1}{(2\pi)^n |\Sigma|} e^{-\frac{1}{2}(\vec{y}-\vec{\mu})\Sigma^{-1}(\vec{y}-\vec{\mu})}$$

must have Σ invertible (non-singular),

and we also require $\text{Var}(\vec{a}^T \vec{Y}) = \vec{a}^T \Sigma \vec{a} > 0$ ^{unless $\vec{a} = 0$}
pos. definite.

A bit strict, can allow

$\vec{a}^T \Sigma \vec{a} \geq 0 \rightarrow$ pos semidef, $f(\vec{y})$ does not exist.

Rep: $\vec{a}^T \vec{Y} \sim N(\vec{a}^T \vec{\mu}, \vec{a}^T \Sigma \vec{a})$ "linear form"

Chap 5: $\vec{Y}^T A \vec{Y}$ "quadratic form" (QF)
 $\uparrow$ $n \times n$ symmetric matrix
and often idempotent $AA = A$

5.2 Mean and variance of QF

Thm 5.2 a)

$\vec{Y}$ random vector, $E(\vec{Y}) = \vec{\mu}$, $\text{Cov}(\vec{Y}) = \Sigma$.

A symmetric. Then

$$E(\vec{Y}^T A \vec{Y}) = \text{tr}(A \Sigma) + \vec{\mu}^T A \vec{\mu}$$

Example: $Y_1, \dots, Y_n$ independent, $Y_i \sim N(\mu, \sigma^2)$ $i=1, \dots, n$.

Estimator for σ^2 :

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2 = \frac{1}{n-1} \vec{Y}^T \left(I - \frac{1}{n} \mathbf{1} \mathbf{1}^T \right) \vec{Y}$$

use trace-formula to show $E(S^2) = \sigma^2$.

$$E(S^2) = \frac{1}{n-1} E(\vec{Y}^T A \vec{Y}) \quad A = \left(I - \frac{1}{n} \mathbf{1} \mathbf{1}^T \right).$$

$$\downarrow \Sigma = \sigma^2 I \quad \vec{\mu} = \mu \vec{j}$$

$$\begin{aligned} & \frac{1}{n-1} \left[\sigma^2 \text{tr} \left(I - \frac{1}{n} \mathbf{1} \mathbf{1}^T \right) + \mu^2 \vec{j}^T \left(I - \frac{1}{n} \mathbf{1} \mathbf{1}^T \right) \vec{j} \right] \\ &= \frac{1}{n-1} \left[\sigma^2 \left(n - \frac{n}{n} \right) + \mu^2 \underbrace{\vec{j}^T (\vec{j} - \vec{j})}_{=0} \right] \\ &= \frac{1}{n-1} \sigma^2 (n-1) = \sigma^2 // \end{aligned}$$

Proof : Use $\text{tr}(A\Sigma) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} \sigma_{ij}$
thm 5.2a)
 $\uparrow \uparrow$
Symmetric

$$\text{Also } \vec{\mu}^T A \vec{\mu} = \sum_{i=1}^n \sum_{j=1}^n a_{ij} \mu_i \mu_j$$

$$\text{add } \rightarrow \sum_{i=1}^n \sum_{j=1}^n a_{ij} (\sigma_{ij} + \mu_i \mu_j)$$

$$\begin{aligned} & \text{Cov}(Y_i, Y_j) + E(Y_i)E(Y_j) \\ &= E(Y_i Y_j) \end{aligned}$$

$$= \sum_{i=1}^n \sum_{j=1}^n a_{ij} E(Y_i Y_j)$$

$$= E\left(\sum_{i=1}^n \sum_{j=1}^n a_{ij} Y_i Y_j\right)$$

$$= E(\vec{Y}^T A \vec{Y})$$

trace formula
just a useful way
of presenting this.

Thm 5.2 c if $\vec{Y} \sim \text{MVN}(\vec{\mu}, \Sigma)$, A symmetric
$$\text{Var}(\vec{Y}^T A \vec{Y}) = 2 \text{tr}[(A \Sigma)^2] + 4 \vec{\mu}^T A \Sigma A \vec{\mu}$$

Will now define 3 (+) different distns.

5.3 χ^2 distribution (chi square)

Def 1: Let $Z_1, \dots, Z_n$ be independent $N(0, 1)$,

$\vec{Z} \sim \text{MVN}_n(\vec{0}, I)$. Then

$$U = \vec{Z}^T \vec{Z} = \sum_{i=1}^n Z_i^2 \sim \chi^2(n)$$

"chi square distribution"

parameter of
 χ^2 -distribution,
"degrees of freedom,
d.o.f".

Thm 5.3a) $U \sim \chi^2(n)$

$$E(U) = n, \quad \text{Var}(U) = 2n,$$

$$(M_U(t) = 1 / (1 - 2t)^{n/2})$$

Proof mean, var.

$$E(\vec{Z}^T \vec{Z}) = \text{tr}(\mathbf{I}\mathbf{I}) + \vec{0}^T \mathbf{I} \vec{0} \\ = n.$$

$$\text{Var}(\vec{Z}^T \vec{Z}) = 2 \text{tr}[(\mathbf{I}\mathbf{I})(\mathbf{I}\mathbf{I})] + 4 \vec{0}^T \mathbf{I} \mathbf{I} \vec{0} \\ = 2n$$

Def 2: Let $Y_1, \dots, Y_n$ be independent $N(\mu_i, \sigma^2)$

$$\vec{Y} \sim \text{MVN}_n(\vec{\mu}, \mathbf{I})$$

"not centered"

$$\text{Then } \vec{Y}^T \vec{Y} = \sum_{i=1}^n Y_i^2 \sim \chi^2(n, \lambda)$$

dof $\uparrow$ noncentrality parameter $\uparrow$

"noncentral chi squared distrn."

$$\lambda = \frac{1}{2} \vec{\mu}^T \vec{\mu} = \frac{1}{2} \sum_{i=1}^n \mu_i^2$$

[Figure 5.1]

Thm 5.3b $V \sim \chi^2(n, \lambda)$

$$E(V) = n + 2\lambda, \quad \text{Var}(V) = 2n + 8\lambda$$

$$\left(M_V(t) = \frac{1}{(1-2t)^{n/2}} e^{-\lambda(1-1/(1-2t))} \right)$$

Proof mean

$$\begin{aligned} E(\vec{Y}^T \vec{Y}) &= \text{tr}(\mathbf{I}\mathbf{I}) + \mu^T \mathbf{I} \mu \\ &= n + \sum_{i=1}^n \mu_i^2 = n + 2\lambda. \end{aligned}$$

NB: $\chi^2(n, 0) = \chi^2(n)$.

non-central χ^2 with non-centrality parameter 0 is central χ^2 .

Note: For a general $Y \sim \text{MVN}(\vec{\mu}, \Sigma)$, we don't have a distrn for $\vec{Y}^T \vec{Y}$. But we have a transformation:

$$\vec{Z} = \Sigma^{-1/2} (\vec{Y} - \vec{\mu}) \sim \text{MVN}(\vec{0}, \vec{\Sigma})$$

Thm 5.3 c) Additive property of χ^2

If $V_1, \dots, V_k$ independent $\chi^2(n_i, \lambda_i)$ then

$$\sum_{i=1}^k V_i \sim \chi^2\left(\sum_{i=1}^k n_i, \sum_{i=1}^k \lambda_i\right)$$

5.4 Noncentral F and t distn.

Def 1: If $U \sim \chi^2(p)$ and $V \sim \chi^2(q)$,
and U, V independent, then

$$\frac{U/p}{V/q} \sim F(p, q)$$

Both parameters called d.o.f.

"F-distribution"

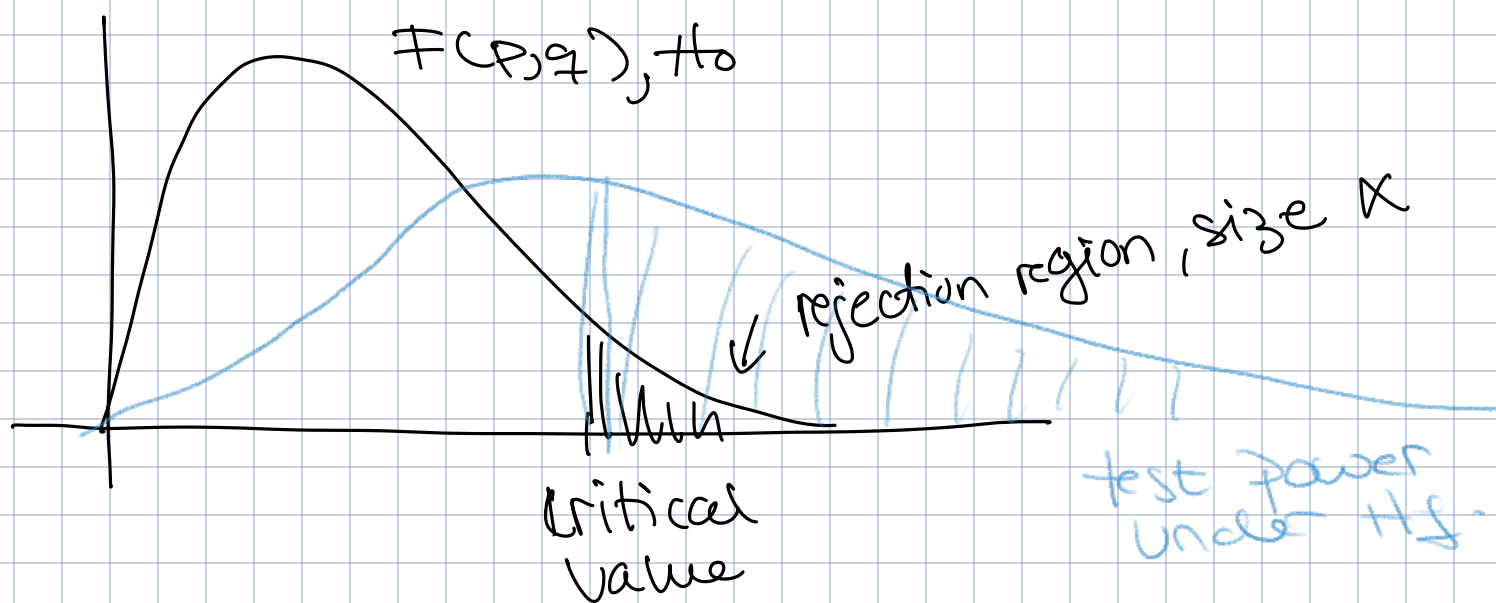
↑
this will represent the distribution
of a test statistic under H_0 .

Def 2: $U \sim \chi^2(p, \lambda)$, $V \sim \chi^2(q)$,
 U, V independent

$$\frac{U/p}{V/q} \sim F(p, q, \lambda)$$

"noncentral F".

distribution
under H_1 ;
alternative
hypothesis.



Def 3: If $Z \sim N(0, 1)$, $U \sim \chi^2(p)$,
 Z, U independent, then

$$\frac{Z}{\sqrt{U/p}} \sim t(p)$$

"t-distribution"

Def 4: If $Y \sim N(\mu, I)$, ← have not centered
 $U \sim \chi^2(p)$, independent

$$\frac{Y}{\sqrt{U/p}} \sim t(p, \mu) \quad \text{"noncentral t-distribution"}$$