

Chap 5 Quadratic forms

Rep: $\vec{Z} \sim \text{MVN}_n(\vec{0}, \mathbf{I})$ (standard normal)

$$\vec{Z}^T \vec{Z} = \sum_{i=1}^n z_i^2 \sim \chi^2(n)$$

$$\vec{Y} \sim \text{MVN}(\vec{\mu}, \Sigma)$$

Mahalanobis: $\Sigma^{-1/2}(\vec{Y} - \vec{\mu}) \sim \text{MVN}(\vec{0}, \mathbf{I})$

$$\Rightarrow (\Sigma^{-1/2}(\vec{Y} - \vec{\mu}))^T (\Sigma^{-1/2}(\vec{Y} - \vec{\mu})) \sim \chi^2(n)$$

$$(\vec{Y} - \vec{\mu})^T \Sigma^{-1/2} \Sigma^{-1/2} (\vec{Y} - \vec{\mu})$$

$$(\vec{Y} - \vec{\mu})^T \Sigma^{-1} (\vec{Y} - \vec{\mu}) \sim \chi^2(n)$$

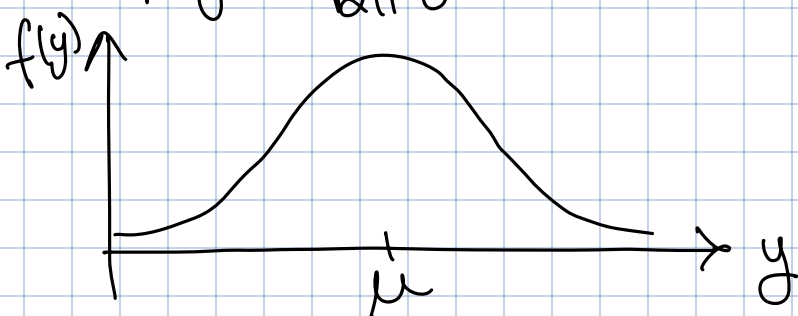
"generalized distance"

why useful?

Example

$$Y \sim N(\mu, \sigma^2)$$

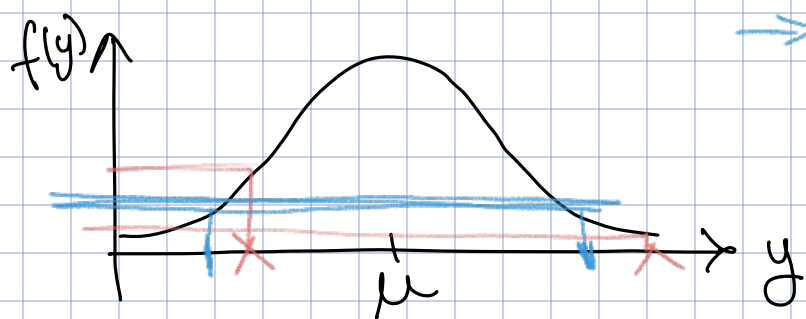
$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \left(\frac{y-\mu}{\sigma}\right)^2}$$



Find an interval $[a, b]$ such that
 $P(a \leq Y \leq b) = 0.95$

± min: When have you done this previously?
 → Confidence intervals.

Note: Did not choose a, b 'randomly',
 but equally far from μ .



→ equal tails, and
 equal $f(a) = f(b)$

→ Restriction $f(a) = f(b)$,
and $P(a \leq Y \leq b) = 0.95$
 ↑ ↑
 $\mu - k$ $\mu + k$
 $k = 1.96 \cdot \sigma$

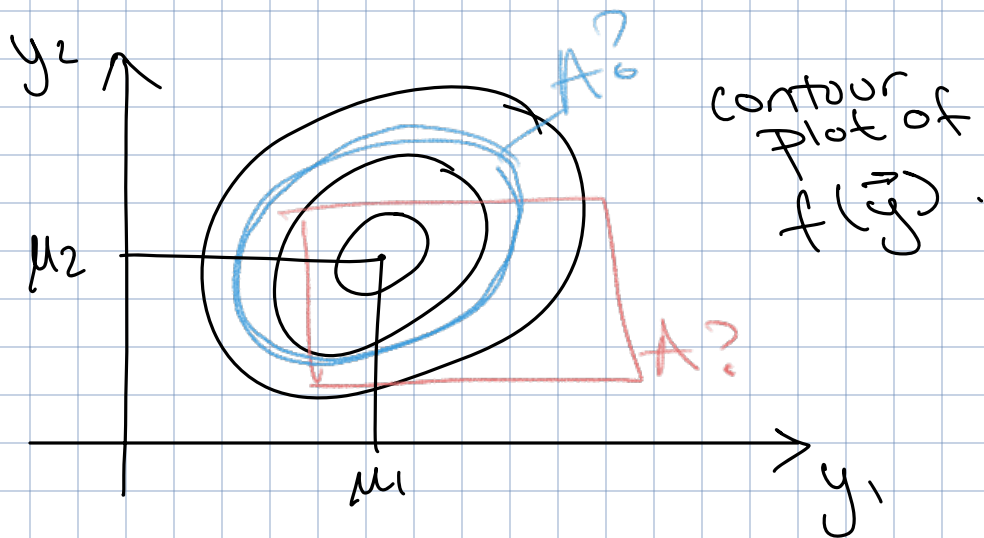
Could also have found k through

$$\begin{aligned}
 P(\mu - k \leq Y \leq \mu + k) &= P\left(-\frac{k}{\sigma} \leq \frac{Y - \mu}{\sigma} \leq \frac{k}{\sigma}\right) \\
 &= P\left(\underbrace{\left(\frac{Y - \mu}{\sigma}\right)^2}_{\chi^2_1} \leq \underbrace{\left(\frac{k}{\sigma}\right)^2}_{3.841}\right) = 0.95 \\
 &\quad \rightarrow k = \sqrt{3.841} \cdot \sigma = 1.96 \cdot \sigma.
 \end{aligned}$$

Bivariate, $\vec{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim \text{MVN}_2(\vec{\mu}, \Sigma)$

Find an area A such that

$$P(\vec{Y} \in A) = 0.95.$$



loc: means
orientation: eigenvectors
half lengths: eigenvalues
of Σ .

At contour: $f(\vec{y})$ is constant
(ellipse)

$$f(\vec{y}) = \frac{1}{\sqrt{(2\pi)^2 |\Sigma|}} e^{-\frac{1}{2} (\vec{y} - \vec{\mu})^T \Sigma^{-1} (\vec{y} - \vec{\mu})}$$

↑
constant
for all $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$
on ellipse.

→

generalized
distance
is constant.

↓
all points on ellipse A
are 'equally far' from $\vec{\mu}$.
 $c > 0$

Find c such that

$$P((\vec{Y} - \vec{\mu})^T \Sigma^{-1} (\vec{Y} - \vec{\mu}) \leq c) = 0.95$$

$\chi^2(n)$, here $n=2 \rightarrow c=5.991$

→ The ellipse defined by all vectors $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ such that

$$f(\vec{y}) = \frac{1}{\sqrt{(2\pi)^2 |\Sigma|}} e^{-\frac{1}{2} \cdot 5.991}$$

gives an area A where $\vec{y} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}$ falls with 95% chance.

5.5 Distribution of quadratic forms $\vec{Y}^T A \vec{Y}$

Thm 5.5 (on slide)

Corollary 1 (to thm 5.5)

$$\vec{Y} \sim \text{MVN}_n(\vec{0}, I),$$

A constant, idempotent, symmetric, $n \times n$,
 $\text{rank}(A) = r$.

$$\text{Then } \vec{Y}^T A \vec{Y} \sim \chi^2(r)$$

Proof: (see book for proof using MGF)

$$\text{We recall } \vec{Z}^T \vec{Z} \sim \chi^2(n)$$

summing
squares of
 n independent
standard
normal.

$$\vec{Y}^T A \vec{Y} \sim \chi^2(r)$$

implies summing $r \leq n$

independent standard normal.

$$(2.103) \quad A = C D C^T$$

$\uparrow \uparrow \text{diag}(\lambda_1, \dots, \lambda_n)$
orthogonal
 $C^T C = C C^T = I$.

$$\vec{Y}^T A \vec{Y} = \vec{Y}^T C D C^T \vec{Y} = (C^T \vec{Y})^T D (C^T \vec{Y})$$

$$C^T \vec{Y} \sim \text{MVN}_n \left(\underset{\substack{\uparrow \\ C^T \vec{\mu}}}{\vec{0}}, \underbrace{C^T I C}_{A \Sigma A^T} \overset{= I}{\quad} \right)$$

$$C^T \vec{Y} = \vec{Z} \sim \text{MVN}(\vec{0}, I).$$

So we are doing something with standard normals.

Thm 2.13c

Eigenvalues in

D are ± 1 (r) or 0 ($n-r$)

$\rightarrow \vec{Z}^T D \vec{Z} = \sum_{i=1}^n \lambda_i z_i^2$ is a sum of r independent standard normal. $\square$.

Corollary 2 $\nwarrow$ not centered

$\vec{Y} \sim \text{MVN}(\vec{\mu}, \sigma^2 I)$, A as before

$$\frac{1}{\sigma^2} \vec{Y} + A \vec{U} \sim \chi^2(r, \frac{\vec{\mu} + A \vec{\mu}}{2\sigma^2}).$$

Example

Recall $Y_1, \dots, Y_n \sim N(\mu, \sigma^2)$ and

$$\text{estimator } S^2 = \frac{1}{n-1} \vec{Y}^T (I - \frac{1}{n} J) \vec{Y}$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

$$\text{Proof: } \frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2} \vec{Y}^T A \vec{Y}$$

with $A = (I - \frac{1}{n} \mathbf{1}\mathbf{1}^T)$, $\text{rank}(A) = n-1$ ✓
 $\vec{Y} \sim \text{MVN}(\vec{\mu}, \sigma^2 I)$ need noncentrality
 $= 0$.

$$\text{in } \lambda: \vec{\mu}^T A \vec{\mu} = (A \vec{\mu})^T (A \vec{\mu})$$

$$A \vec{\mu} = (I - \frac{1}{n} \mathbf{1}\mathbf{1}^T) \vec{\mu} = \vec{0}.$$

Also, $\bar{Y}$ estimator for μ , $\bar{Y} = \frac{1}{n} \mathbf{1}^T \vec{Y}$.

Test $H_0: \mu = \mu_0$

$$T = \frac{\bar{Y} - \mu_0}{s/\sqrt{n}} = \frac{\bar{Y} - \mu_0}{\frac{\sigma/\sqrt{n}}{\sqrt{\frac{(n-1)S^2}{\sigma^2} / (n-1)}}}$$

Recall:
t-distr if
standard
normal
and χ^2
independent.

Need $\bar{Y}$ and s^2 independent.

5.6 Independence of linear and quadratic forms

Thm 5.6 a) B $k \times n$
 A $n \times n$ sym. idempotent
 $\vec{Y} \sim MVN(\vec{\mu}, \Sigma)$

$B\vec{Y}$ and $\vec{Y}^T A \vec{Y}$ independent
iff $B\Sigma A = 0$.

Example, cont:

$$\bar{Y} = \underbrace{\frac{1}{n} \vec{1}^T \vec{Y}}_B$$

$$s^2 = \underbrace{\vec{Y}^T (I - \frac{1}{n} \vec{1} \vec{1}^T) \vec{Y}}_A$$

$$\Sigma = \sigma^2 I$$

$$B\Sigma A = \frac{1}{n} \vec{1}^T \sigma^2 I (I - \frac{1}{n} \vec{1} \vec{1}^T) = \vec{0}.$$

Proof: $\vec{Y}^T A \vec{Y} = \vec{Y}^T A A \vec{Y} = (A \vec{Y})^T (A \vec{Y})$

Only need to show $A\vec{Y}$, $B\vec{Y}$ independent,
follows from ind- of linear forms.

Thm 5.6 b

A, B sym. idempotent

$\vec{y}^T A \vec{y}$, $\vec{y}^T B \vec{y}$ independent

iff $AB = 0$