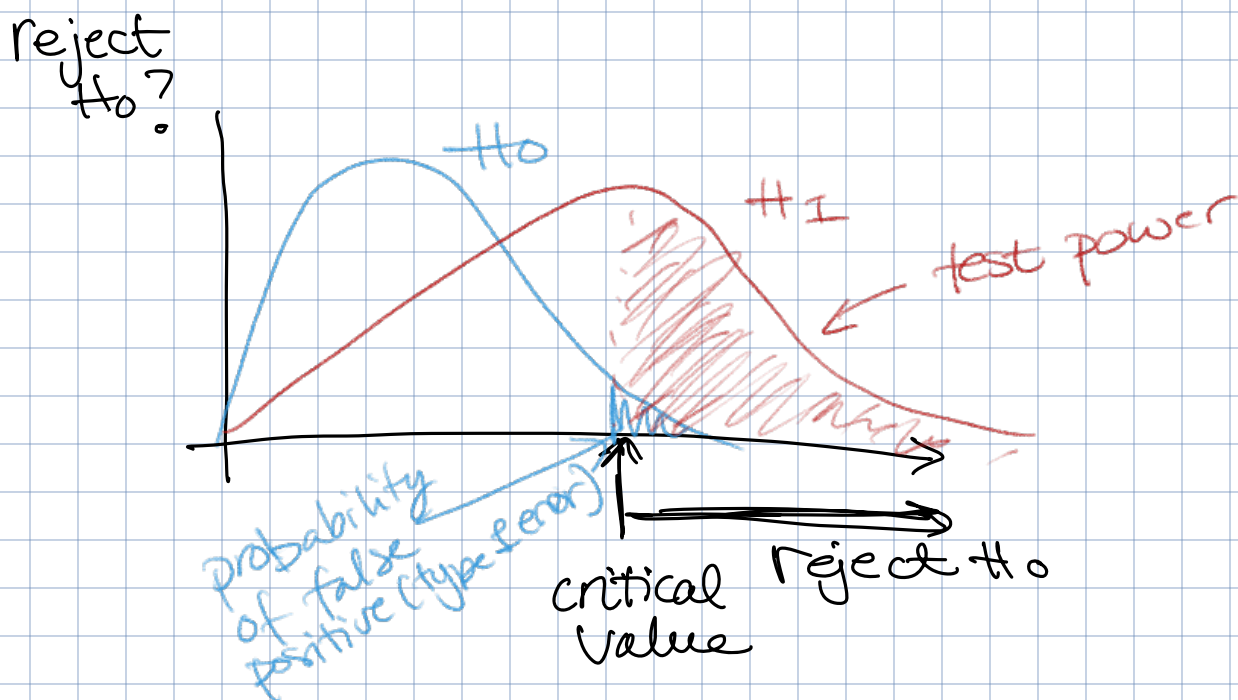


Chap 8 Tests, confidence intervals

Rep: $Y = \beta_0 + X_1 \beta_1 + \varepsilon$

$H_0: \vec{\beta}_1 = \vec{0}$, $H_1: \vec{\beta}_1 \neq \vec{0}$ — at least one β diff.

$$F = \frac{SSR/k}{SSE/(n-k-1)} \sim F(k, n-k-1)$$



2min: How to choose critical value?

- Set $P(\text{reject } H_0 | H_0 \text{ true}) = \alpha$
e.g. $\alpha = 0.05$
- Reject if $F_{\text{obs}} > F_\alpha$ where $P(F > F_\alpha | H_0) = \alpha$.

or equivalently $p\text{-value} < \alpha$

$$p\text{-value} = P(F > F_{\text{obs}} | H_0)$$

→ More F-tests

Thm 8.2c Partial F-test

$$\vec{\beta} = \begin{pmatrix} \vec{\beta}_1 \\ \vec{\beta}_2 \end{pmatrix} \quad H_0: \vec{\beta}_2 = \vec{0}$$

↑ length h

$$X = [X_1 \ X_2]$$

$$H = X(X^T X)^{-1} X^T$$

(full model)

$$H^* = X_1(X_1^T X_1)^{-1} X_1^T$$

(reduced model)

$$SSE = \vec{Y}^T (I - H) \vec{Y} \quad SSE^* = \vec{Y}^T (I - H^*) \vec{Y}$$

$$F = \frac{(SSE^* - SSE)/h}{SSE/(n-k-1)} \quad \overset{H_0}{\sim} F(h, n-k-1)$$

(noncentral, H_1)

Thm 8.4g General linear hypothesis test

$$H_0: C\vec{\beta} = \vec{t}$$

↑
 $q \times (k+1)$
 $\text{rank}(C) = q$

$$\text{Ex: } C = \begin{bmatrix} \vec{0} & I_k \end{bmatrix} \quad \vec{t} = \vec{0}$$

gives F test overall model

$$C = \begin{bmatrix} 0 & I_h \end{bmatrix} \quad \vec{t} = \vec{0}$$

partial F test

Use $\hat{C\beta} \sim \text{MVN}(C\vec{\beta}, \sigma^2 C(X^T X)^{-1} C^T)$

$$\vec{Z} = \text{cov}(\hat{C\beta})^{-1/2} (C\vec{\beta} - \vec{t}) \overset{H_0}{\sim} \text{MVN}(\vec{0}, I)$$

(mahalanobis)

$$\vec{Z}^T \vec{Z} \stackrel{H_0}{\sim} \chi^2(q)$$

$$F = \frac{(C\hat{\beta} - \vec{t})^T (C(X^T X)^{-1} C^T)^{-1} (C\hat{\beta} - \vec{t})}{SSE / (n - k - 1)} \stackrel{H_0}{\sim} F(q, n - k - 1)$$

recall σ^2

8.5.1

$$H_0: \beta_j = 0 \quad H_1: \beta_j \neq 0$$

$$C = [0 \dots 0 \ 1 \ 0 \dots 0] \quad t = 0$$

$$F = \frac{\hat{\beta}_j (g_{jj})^{-1} \hat{\beta}_j}{\sigma^2} \quad g_{jj} \text{ jth element of } (X^T X)^{-1}$$

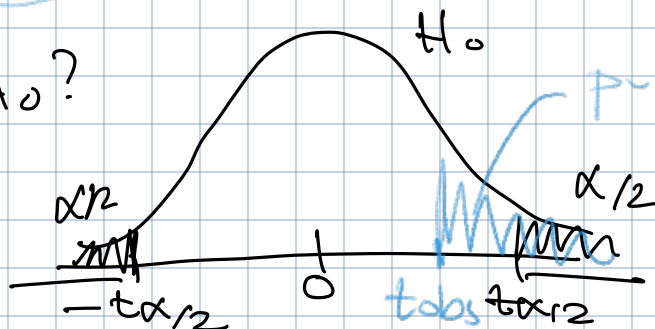
$$= \frac{\hat{\beta}_j^2}{\sigma^2 g_{jj}} \stackrel{H_0}{\sim} F(1, n - k - 1)$$

↓

$$T = \frac{\hat{\beta}_j}{s \sqrt{g_{jj}}} \stackrel{H_0}{\sim} t(n - k - 1).$$

read both from r.

Reject H_0 ?



R example

Use model = central
as example.

$[R = \text{confint} \dots]$ ← no interval includes 0.

8.6 Confidence intervals.

$(1-\alpha)$ 100% Conf. int for θ ($\alpha=0.05 \Rightarrow 95\%$)

is $[L, U]$ such that $P(L \leq \theta \leq U) = 1-\alpha$
↑ ↑
Stochastic

8.6.3 β_j

$$\hat{\beta} = (X^T X)^{-1} X^T \vec{y} \sim \text{MVN}_{k+1}(\vec{\beta}, \sigma^2 (X^T X)^{-1})$$

then,

$$\hat{\beta}_j \sim N(\beta_j, \sigma^2 g_{jj})$$

$$\frac{\hat{\beta}_j - \beta_j}{\sigma \sqrt{g_{jj}}} \sim N(0,1) \dots \rightarrow \frac{\hat{\beta}_j - \beta_j}{\frac{\sigma \sqrt{g_{jj}}}{\sqrt{\frac{(n-k-1)s^2}{(n-k-1)}}}} \sim t(n-k-1)$$

$N(0,1)$

$$\Rightarrow \frac{\hat{\beta}_j - \beta_j}{s \sqrt{g_{jj}}} \sim t(n-k-1)$$

$\frac{\chi^2(n-k-1)}{\text{dof}}$
 $s^2, \hat{\beta}_j$ independent

$$P(-t_{\alpha/2} \leq \frac{\hat{\beta}_j - \beta_j}{s \sqrt{g_{jj}}} \leq t_{\alpha/2}) = 1-\alpha$$

Stochastic

!

$$P(\hat{\beta}_j - t_{\alpha/2} S \sqrt{g_{jj}} \leq \beta_j \leq \hat{\beta}_j + t_{\alpha/2} S \sqrt{g_{jj}}) = 1 - \alpha.$$

NB : Confidence ellipse (region) for multivariate normal $\hat{\beta}$, see R.

8.6.4 $E(Y)$

Select covariates $\vec{X}_0$, then

$$E(Y_0) = \vec{X}_0^T \vec{\beta}$$

$$\hat{Y}_0 = E(\hat{Y}_0) = \vec{X}_0^T \hat{\beta} \sim N(\vec{X}_0^T \vec{\beta}, \sigma^2 \vec{X}_0^T (X^T X)^{-1} \vec{X}_0)$$

$(1-\alpha)100\%$ conf. int for $E(Y_0)$:

$$\vec{X}_0^T \hat{\beta} \pm t_{\alpha/2} S \sqrt{\vec{X}_0^T (X^T X)^{-1} \vec{X}_0}$$

R example.

8.6.5 Prediction interval Y_0

$$Y_0 = \vec{X}_0^T \vec{\beta} + \varepsilon_0$$

We have $\hat{\vec{\beta}}$ and S^2 from data.

ind. $\uparrow$

$$Y_0 \sim N(\vec{X}_0^T \vec{\beta}, \sigma^2)$$
$$\hat{Y}_0 \sim N(\vec{X}_0^T \hat{\vec{\beta}}, \sigma^2 \vec{X}_0^T (X^T X)^{-1} \vec{X}_0)$$

$$Y_0 - \hat{Y}_0 \sim N(0, \sigma^2 (1 + \vec{X}_0^T (X^T X)^{-1} \vec{X}_0))$$

$(1-\alpha)100\%$ pred. int. for Y_0 :

$$\hat{Y}_0 \pm t_{\alpha/2} S \sqrt{1 + \vec{X}_0^T (X^T X)^{-1} \vec{X}_0}$$