Proof of mean of estimator for $Cov(\underline{X})$

Background:

$\underset{p\times 1}{\underline{X}}$ has mean $E(\underline{X}) = \underset{p\times 1}{\mu}$ and $Cov(\underline{X}) = \underset{p\times p}{\Sigma}$

$\underline{X}_1, \dots, \underline{X}_n$ is a random sample from $\underline{X}$.

$$\overline{\underline{X}} = \frac{1}{n} \sum_{j=1}^{n} \underline{X}_j$$

Estimator:

$$S = \frac{1}{n-1} \sum_{j=1}^{n} (\underline{X}_j - \overline{\underline{X}})(\underline{X}_j - \overline{\underline{X}})^T$$

Result:

$E(S) = \underset{p\times p}{\Sigma}$ unbiased estimator

Proof:

i) $\sum_{j=1}^{n} (\underline{X}_j - \overline{\underline{X}})(\underline{X}_j - \overline{\underline{X}})^T$

$$= \sum_{j=1}^{n} (\underline{X}_j - \overline{\underline{X}})\underline{X}_j^T - \underbrace{\sum_{j=1}^{n} (\underline{X}_j - \overline{\underline{X}})\overline{\underline{X}}^T}_{0}$$

because $\underbrace{\sum_{j=1}^{n} X_j \bar{X}^T}$ $- \sum_{j=1}^{n} \bar{X}\bar{X}^T$

$\frac{1}{n}\sum_{j=1}^{n} X_j = \bar{X}$

$n\bar{X} = \sum_{j=1}^{n} X_j$

$n\bar{X}\bar{X}^T - n\bar{X}\bar{X}^T = 0$

$$\sum_{j=1}^{n} (X_j - \bar{X})X_j^T = \sum_{j=1}^{n} X_j X_j^T - \sum_{j=1}^{n} \bar{X} X_j^T$$

$$= \sum_{j=1}^{n} X_j X_j^T - n\bar{X}\bar{X}^T$$

so $E(S) = \frac{1}{n-1} E\left[\sum_{j=1}^{n} X_j X_j^T - n\bar{X}\bar{X}^T\right]$

$$= \frac{1}{n-1}\left\{\sum_{j=1}^{n} E(X_j X_j^T) - n E(\bar{X}\bar{X}^T)\right\}$$

ii) Know: $\mathrm{Cov}(V) = E(VV^T) - E(V)E(V)^T$

$E(VV^T) = \mathrm{Cov}(V) + E(V)E(V)^T$

so we need to use

$$E(X_j X_j^T) = \mathrm{Cov}(X_j) + E(X_j)E(X_j)^T$$
$$= \Sigma + \mu\mu^T$$

and

$$E(\bar{X}\bar{X}^T) = \mathrm{Cov}(\bar{X}) + E(\bar{X})E(\bar{X})^T$$
$$= \frac{1}{n}\Sigma + \mu\mu^T$$

so connecting i) and ii)

$$E(S) = \frac{1}{n-1}\left\{ \sum_{j=1}^{n} E(\mathbf{X}_j \mathbf{X}_j^T) - n\, E(\bar{\mathbf{X}}\bar{\mathbf{X}}^T) \right\}$$

$$= \frac{1}{n-1}\left\{ \sum_{j=1}^{n} \left(\Sigma + \mu\mu^T\right) - n\left(\frac{1}{n}\Sigma + \mu\mu^T\right) \right\}$$

$$= \frac{1}{n-1}\left\{ n\Sigma + n\mu\mu^T - \Sigma - n\mu\mu^T \right\}$$

$$= \frac{1}{n-1}\left\{ n\Sigma - \Sigma \right\} = \frac{n-1}{n-1}\Sigma = \underline{\underline{\Sigma}}$$