Properties of $\hat{\beta}$

$$\hat{\beta} = \underbrace{(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T}_{\substack{C \\ \| \\ \text{constants}}} \underset{\substack{\uparrow \\ \text{RV}}}{Y} \qquad \text{and} \qquad E(Y) = \mathbb{X}\beta$$

$$\mathrm{Cov}(Y) = \sigma^2 I$$

$$Y = \mathbb{X}\beta + \varepsilon \quad E(\varepsilon) = 0, \ \mathrm{Cov}(\varepsilon) = \sigma^2 I$$

Find $E(\hat{\beta})$ and $\mathrm{Cov}(\hat{\beta})$:

$$E(\hat{\beta}) = E(CY) = C \underset{\substack{\| \\ \mathbb{X}\beta}}{E(Y)} = \underbrace{(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\mathbb{X}}_{I}\beta = \underline{\underline{\beta}}$$

unbiased

$$\mathrm{Cov}(\hat{\beta}) = \mathrm{Cov}(CY) = C \underbrace{\mathrm{Cov}(Y)}_{\sigma^2 I} C^T$$

$$= (\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T \sigma^2 I \left[(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\right]^T$$

$$= \sigma^2 (\mathbb{X}^T\mathbb{X})^{-1} \underbrace{\mathbb{X}^T\mathbb{X}(\mathbb{X}^T\mathbb{X})^{-1}}_{I}$$

$\mathbb{X}^T\mathbb{X}$ is symmetric
$(\mathbb{X}^T\mathbb{X})^{-1}$ — " —

$$= \underline{\underline{\sigma^2 (\mathbb{X}^T\mathbb{X})^{-1}}}$$