LECTURE WEEK 8
Spring 2005
March 15 and 18

TMA4275 LIFETIME ANALYSIS

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COMPUTER PROGRAM EXECUTION TIME vs SYSTEM LOAD

Data: 17 observations of (T,x)

- **Time to complete a computationally intensive task.**
- **Information from the Unix uptime command**
- **Predictions needed for scheduling subsequent steps in a multi-step computational process.**

Seconds (T)	Load (x)	Seconds (T)	Load (x)
123	2,74	110	,60
704	5,47	213	2,10
184	2,13	284	3,10
113	1,00	317	5,86
94	,32	142	1,18
76	,31	127	,57
78	,51	96	1,10
98	,29	111	1,89
240	,96		

Covariates (explanatory variables) for failure times

Useful covariates explain/predict why some units fail quickly and some units survive a long time:

- Continuous variables like stress, temperature, voltage, and pressure.
- Discrete variables like number of hardening treatments or number of simultaneous users of a system.
- Categorical variables like manufacturer, design, and location.

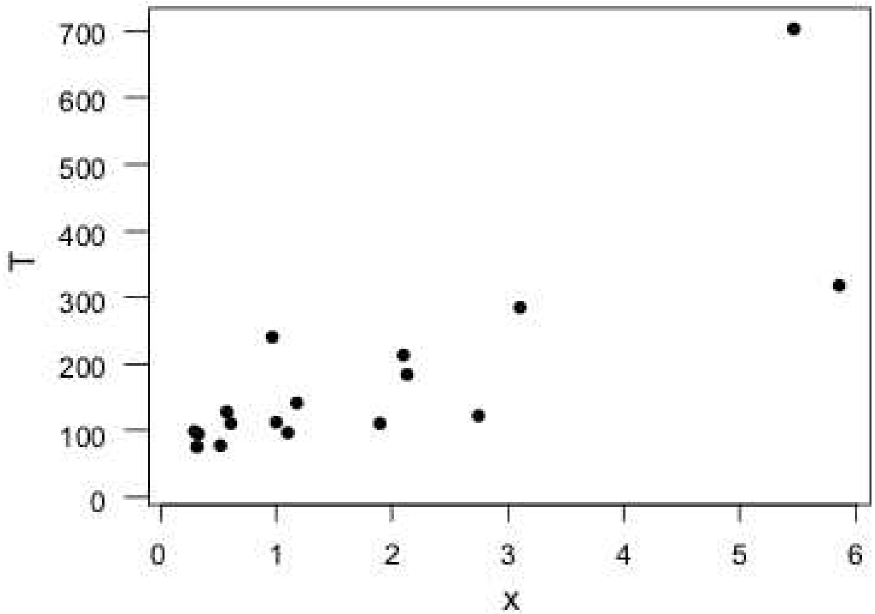
Regression model relates failure time distribution to covariates $x = (x_1, \ldots, x_k)$:

$$P(T \leq t) = F(t) = F(t; x)$$

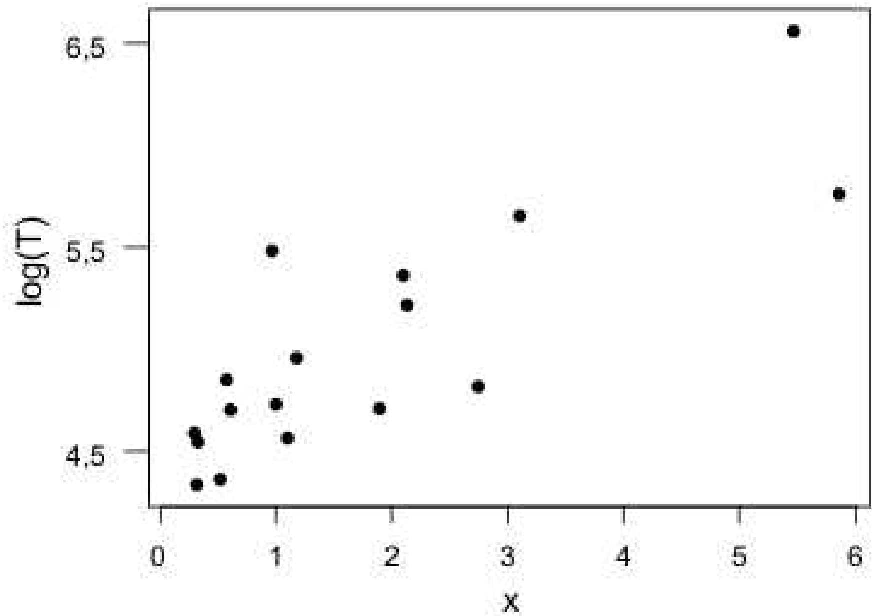
Why regression models?

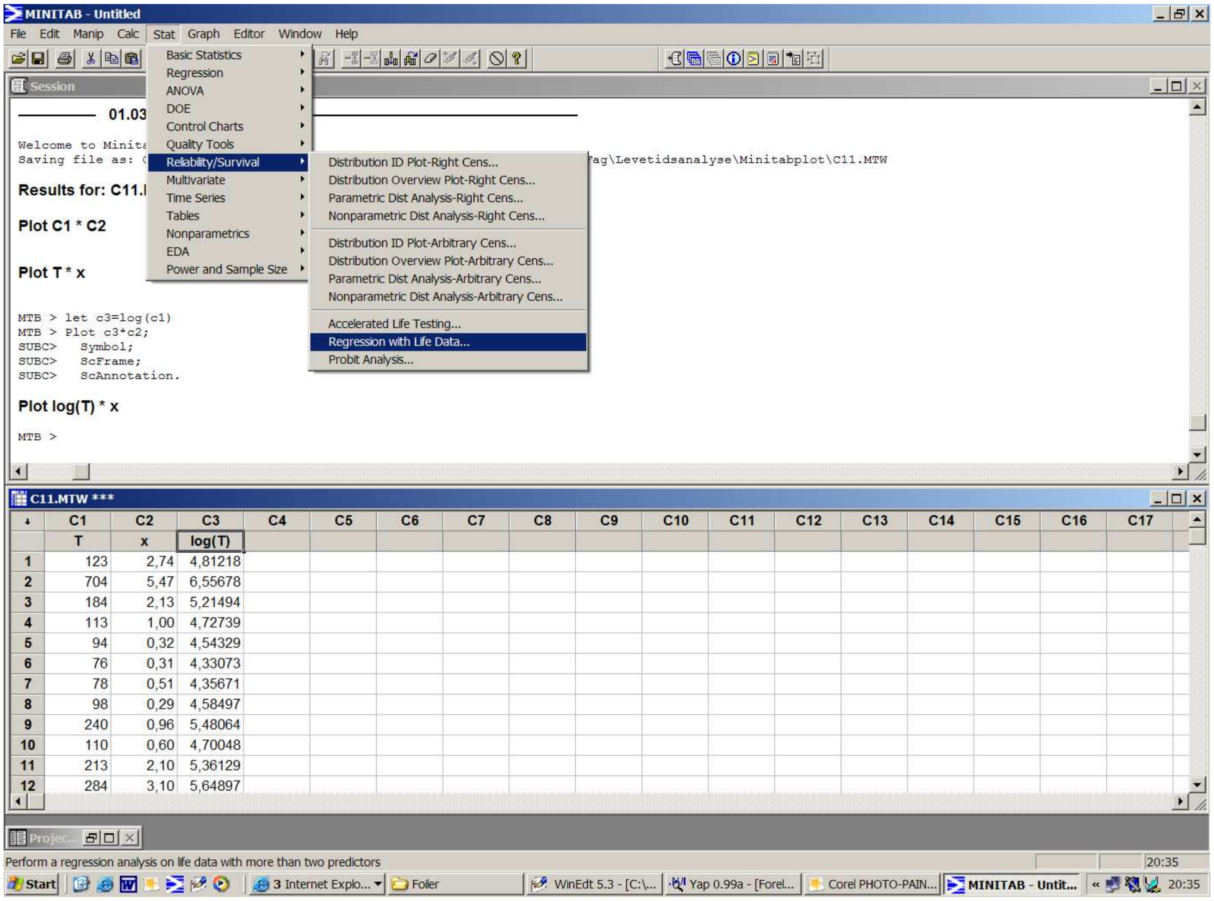
- Want to find factors which explain the reliability of an item
- Want to exclude factors which do not influence the reliability
- Obtain new knowledge about failure mechanisms
- Make better predictions for reliability of an item

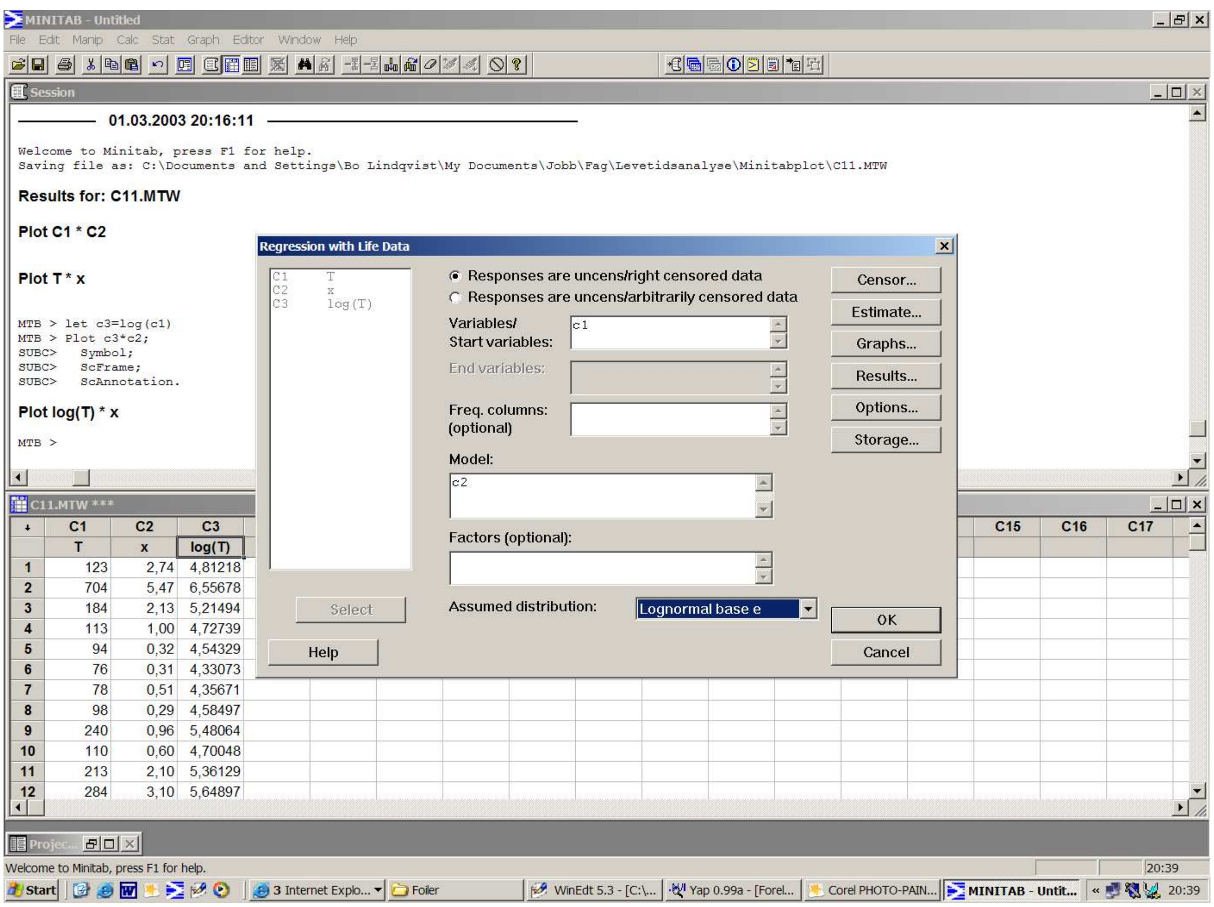
Computer data



Computer data




MINITAB - Untitled
File Edit Manip Calc Stat Graph Editor Window Help
Session
Basic Statistics
Regression
ANOVA
DOE
Control Charts
Quality Tools
Reliability/Survival
Multivariate
Time Series
Tables
Nonparametrics
EDA
Power and Sample Size
Distribution ID Plot-Right Cens...
Distribution Overview Plot-Right Cens...
Parametric Dist Analysis-Right Cens...
Nonparametric Dist Analysis-Right Cens...
Distribution ID Plot-Arbitrary Cens...
Distribution Overview Plot-Arbitrary Cens...
Parametric Dist Analysis-Arbitrary Cens...
Nonparametric Dist Analysis-Arbitrary Cens...
Accelerated Life Testing...
Regression with Life Data...
Probit Analysis...
Results for: C11.
Plot C1 * C2
Plot T * x
MTB > let c3=log(c1)
MTB > Plot c3*c2;
SUBC> Symbol;
SUBC> ScFrame;
SUBC> ScAnnotation.
Plot log(T) * x
MTB >
C11.MTW ***
C1 C2 C3
T x log(T)
1 123 2,74 4,81218
2 704 5,47 6,55678
3 184 2,13 5,21494
4 113 1,00 4,72739
5 94 0,32 4,54329
6 76 0,31 4,33073
7 78 0,51 4,35671
8 98 0,29 4,58497
9 240 0,96 5,48064
10 110 0,60 4,70048
11 213 2,10 5,36129
12 284 3,10 5,64897
Perform a regression analysis on life data with more than two predictors


MINITAB - Untitled
Session
01.03.2003 20:16:11
Welcome to Minitab, press F1 for help.
Saving file as: C:\Documents and Settings\Bo Lindqvist\My Documents\Jobb\Fag\Levetidsanalyse\Minitabplot\C11.MTW
Results for: C11.MTW
Plot C1 * C2
Plot T * x
MTB > let c3=log(c1)
MTB > Plot c3*c2;
SUBC> Symbol;
SUBC> ScFrame;
SUBC> ScAnnotation.
Plot log(T) * x
MTB >
Regression with Life Data
C1 T
C2 x
C3 log(T)
Responses are uncens/right censored data
Responses are uncens/arbitrarily censored data
Variables/
Start variables:
c1
End variables:
Freq. columns:
(optional)
Model:
c2
Factors (optional):
Assumed distribution:
Lognormal base e
Censor...
Estimate...
Graphs...
Results...
Options...
Storage...
Select
Help
OK
Cancel
C11.MTW ***
C1 C2 C3
T x log(T)
1 123 2,74 4,81218
2 704 5,47 6,55678
3 184 2,13 5,21494
4 113 1,00 4,72739
5 94 0,32 4,54329
6 76 0,31 4,33073
7 78 0,51 4,35671
8 98 0,29 4,58497
9 240 0,96 5,48064
10 110 0,60 4,70048
11 213 2,10 5,36129
12 284 3,10 5,64897
Welcome to Minitab, press F1 for help.

Regression with Life Data: T versus x

```
Response Variable:  T

Censoring Information              Count
Uncensored value                      17

Estimation Method:  Maximum Likelihood
Distribution:  Lognormal base e

Regression Table
                       Standard                          95,0% Normal CI
Predictor        Coef     Error        Z      P        Lower       Upper
Intercept      4,4936    0,1112    40,39  0,000       4,2756      4,7116
x             0,29075   0,04595     6,33  0,000      0,20069     0,38080
Scale         0,31247   0,05359                      0,22327     0,43730

Log-Likelihood = -89,498

Anderson-Darling (adjusted) Goodness-of-Fit

Standardized Residuals = 0,8356; Cox-Snell Residuals = 0,8170
```


Regression with Life Data: C1 versus C2

```
Response Variable:  C1

Censoring Information              Count
Uncensored value                      17

Estimation Method:  Maximum Likelihood
Distribution:  Weibull

Regression Table
                       Standard                          95,0% Normal CI
Predictor        Coef     Error        Z      P        Lower       Upper
Intercept      4,6182    0,1219    37,88  0,000       4,3792      4,8572
C2            0,31118   0,04939     6,30  0,000      0,21437     0,40799
Shape          3,0604    0,5245                       2,1873      4,2820

Log-Likelihood = -91,504

Anderson-Darling (adjusted) Goodness-of-Fit
```

Lognormal Distribution Simple Regression Model with Constant Shape Parameter $\beta = 1/\sigma$

- The lognormal simple regression model is

$$\Pr(T \le t) = F(t; \mu, \sigma) = F(t; \beta_0, \beta_1, \sigma) = \Phi_{\text{nor}}\left[\frac{\log(t) - \mu}{\sigma}\right]$$

where $\mu = \mu(x) = \beta_0 + \beta_1 x$ and σ does not depend on x.

- The failure-time log quantile function

$$\log[t_p(x)] = \mu(x) + \Phi_{\text{nor}}^{-1}(p)\sigma$$

is linear in x.

Notice that

$$\frac{t_p(x)}{t_p(0)} = \exp\left(\beta_1 x\right)$$

implies that this regression model is a scale accelerated failure time (SAFT) model with $\mathcal{AF}(x) = \exp(-\beta_1 x)$.

Likelihood for Lognormal Distribution Simple Regression Model with Right Censored Data

The likelihood for n independent observations has the form

$$\begin{aligned} L(\beta_0, \beta_1, \sigma) &= \prod_{i=1}^{n} L_i(\beta_0, \beta_1, \sigma; \text{data}_i) \\ &= \prod_{i=1}^{n} \left\{\frac{1}{\sigma t_i}\phi_{\text{nor}}\left[\frac{\log(t_i) - \mu_i}{\sigma}\right]\right\}^{\delta_i} \left\{1 - \Phi_{\text{nor}}\left[\frac{\log(t_i) - \mu_i}{\sigma}\right]\right\}^{1-\delta_i} \end{aligned}$$

where $\text{data}_i = (x_i, t_i, \delta_i)$, $\mu_i = \beta_0 + \beta_1 x_i$,

$$\delta_i = \begin{cases} 1 & \text{exact observation} \\ 0 & \text{right censored observation} \end{cases}$$

$\phi_{\text{nor}}(z)$ is the standardized normal pdf and $\Phi_{\text{nor}}(z)$ is the corresponding normal cdf.

The parameters are $\boldsymbol{\theta} = (\beta_0, \beta_1, \sigma)$.

Estimated Parameter Variance-Covariance Matrix

Local (observed information) estimate

$$\widehat{\Sigma}_{\widehat{\boldsymbol{\theta}}} = \begin{bmatrix} \widehat{\text{Var}}(\widehat{\beta}_0) & \widehat{\text{Cov}}(\widehat{\beta}_0, \widehat{\beta}_1) & \widehat{\text{Cov}}(\widehat{\beta}_0, \widehat{\sigma}) \\ \widehat{\text{Cov}}(\widehat{\beta}_1, \widehat{\beta}_0) & \widehat{\text{Var}}(\widehat{\beta}_1) & \widehat{\text{Cov}}(\widehat{\beta}_1, \widehat{\sigma}) \\ \widehat{\text{Cov}}(\widehat{\sigma}, \widehat{\beta}_0) & \widehat{\text{Cov}}(\widehat{\sigma}, \widehat{\beta}_1) & \widehat{\text{Var}}(\widehat{\sigma}) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\partial^2 \mathcal{L}(\beta_0,\beta_1,\sigma)}{\partial \beta_0^2} & -\frac{\partial^2 \mathcal{L}(\beta_0,\beta_1,\sigma)}{\partial \beta_0 \partial \beta_1} & -\frac{\partial^2 \mathcal{L}(\beta_0,\beta_1,\sigma)}{\partial \beta_0 \partial \sigma} \\ -\frac{\partial^2 \mathcal{L}(\beta_0,\beta_1,\sigma)}{\partial \beta_1 \partial \beta_0} & -\frac{\partial^2 \mathcal{L}(\beta_0,\beta_1,\sigma)}{\partial \beta_1^2} & -\frac{\partial^2 \mathcal{L}(\beta_0,\beta_1,\sigma)}{\partial \beta_1 \partial \sigma} \\ -\frac{\partial^2 \mathcal{L}(\beta_0,\beta_1,\sigma)}{\partial \sigma \partial \beta_0} & -\frac{\partial^2 \mathcal{L}(\beta_0,\beta_1,\sigma)}{\partial \sigma \partial \beta_1} & -\frac{\partial^2 \mathcal{L}(\beta_0,\beta_1,\sigma)}{\partial \sigma^2} \end{bmatrix}^{-1}$$

Partial derivatives are evaluated at $\widehat{\beta}_0, \widehat{\beta}_1, \widehat{\sigma}$.

Standard Errors and Confidence Intervals for Parameters

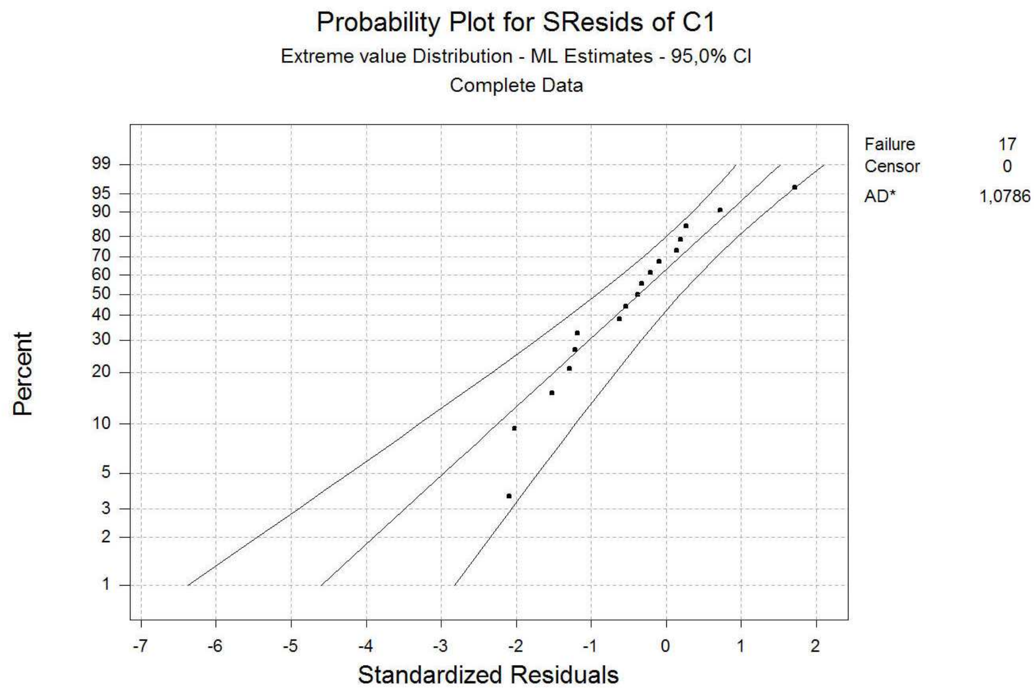
- Lognormal ML estimates for the computer time experiment were $\widehat{\boldsymbol{\theta}} = (\widehat{\beta}_0, \widehat{\beta}_1, \widehat{\sigma}) = (4.49, .290, .312)$ and an estimate of the variance-covariance matrix for $\widehat{\boldsymbol{\theta}}$ is

$$\widehat{\Sigma}_{\widehat{\boldsymbol{\theta}}} = \begin{bmatrix} .012 & -.0037 & 0 \\ -.0037 & .0021 & 0 \\ 0 & 0 & .0029 \end{bmatrix}.$$

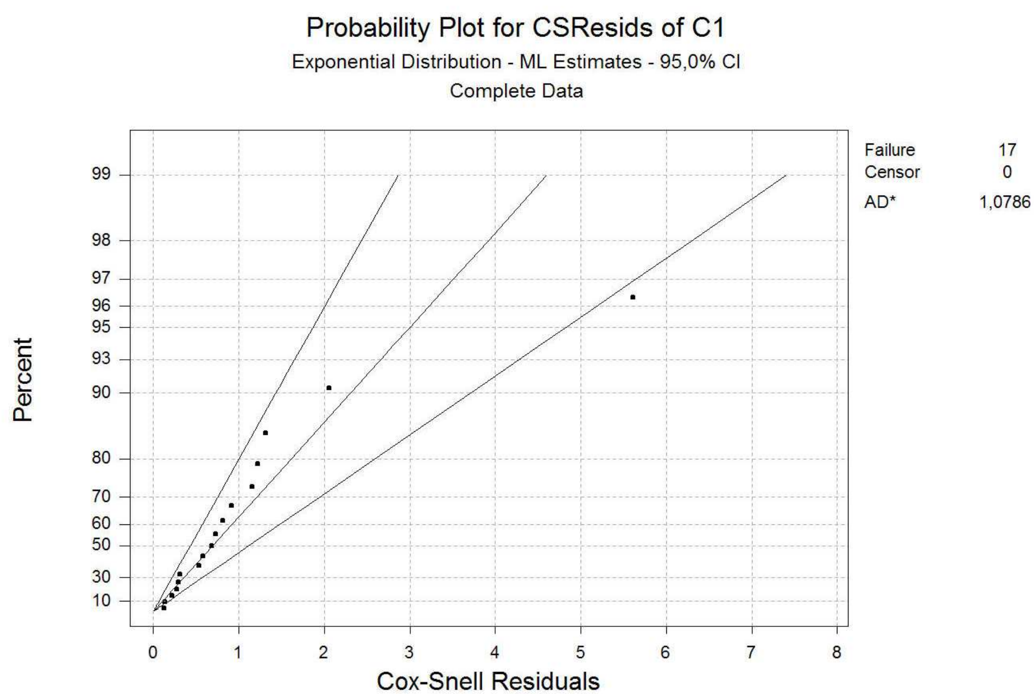
- Normal-approximation confidence interval for the computer execution time regression slope is

$$[\underset{\sim}{\beta_1}, \quad \widetilde{\beta}_1] = \widehat{\beta}_1 \pm z_{(.975)} \widehat{\text{se}}_{\widehat{\beta}_1} = .290 \pm 1.96(.046) = [.20, \quad .38]$$

where $\widehat{\text{se}}_{\widehat{\beta}_1} = \sqrt{.0021} = .046$.



Probability Plot for SResids of C1
Extreme value Distribution - ML Estimates - 95,0% CI
Complete Data
Percent
Standardized Residuals
Failure 17
Censor 0
AD* 1,0786



Probability Plot for CSResids of C1
Exponential Distribution - ML Estimates - 95,0% CI
Complete Data
Percent
Cox-Snell Residuals
Failure 17
Censor 0
AD* 1,0786

Ordinary residuals

$y_i - x'_i\hat{\beta}$

where

yi is the ith response value

x'_i is the vector of predictor values associated with the ith response value

$\hat{\beta}$ represents the estimated regression coefficients

Standardized residuals

$$\frac{y_i - x'_i\hat{\beta}}{\hat{\sigma}}$$

where $\hat{\sigma}$ is the estimated scale parameter.

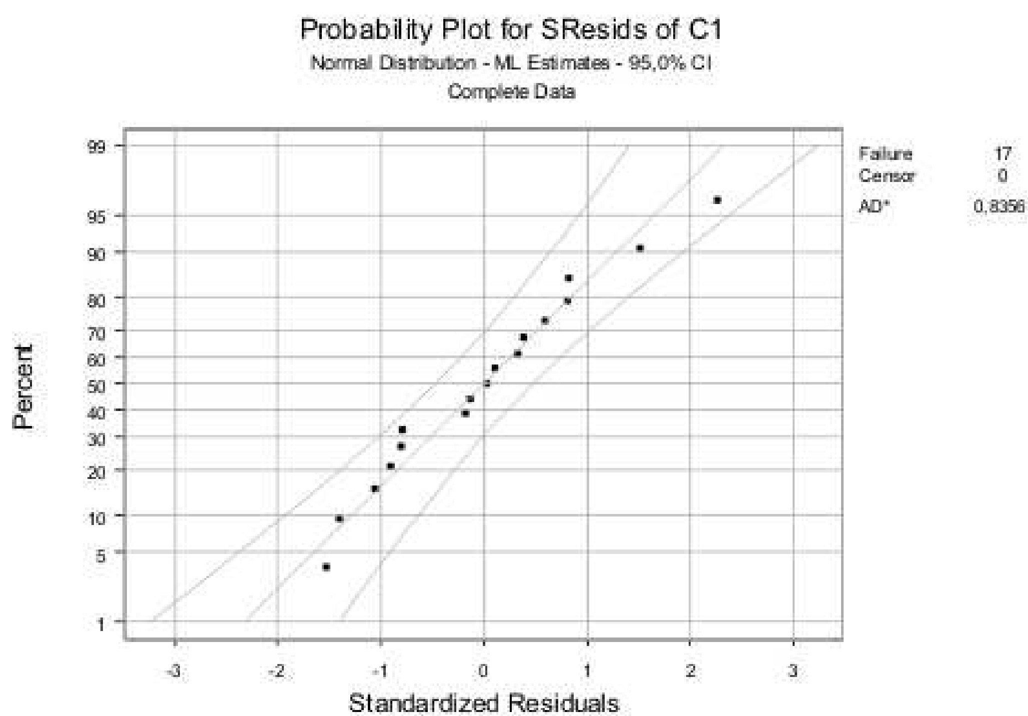
Cox-Snell residuals

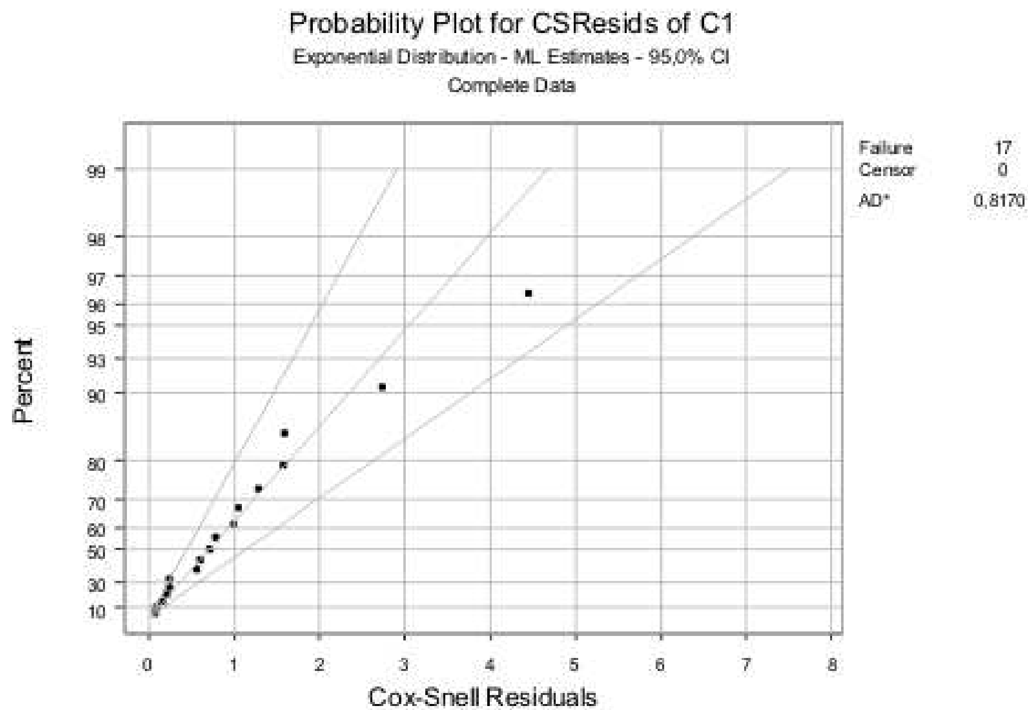
$-\ln(\hat{R}(y_i))$

where

$\hat{R}(y_i)$ is the estimated survival (reliability) probability for the response value yi

ln(x) is the natural log of x

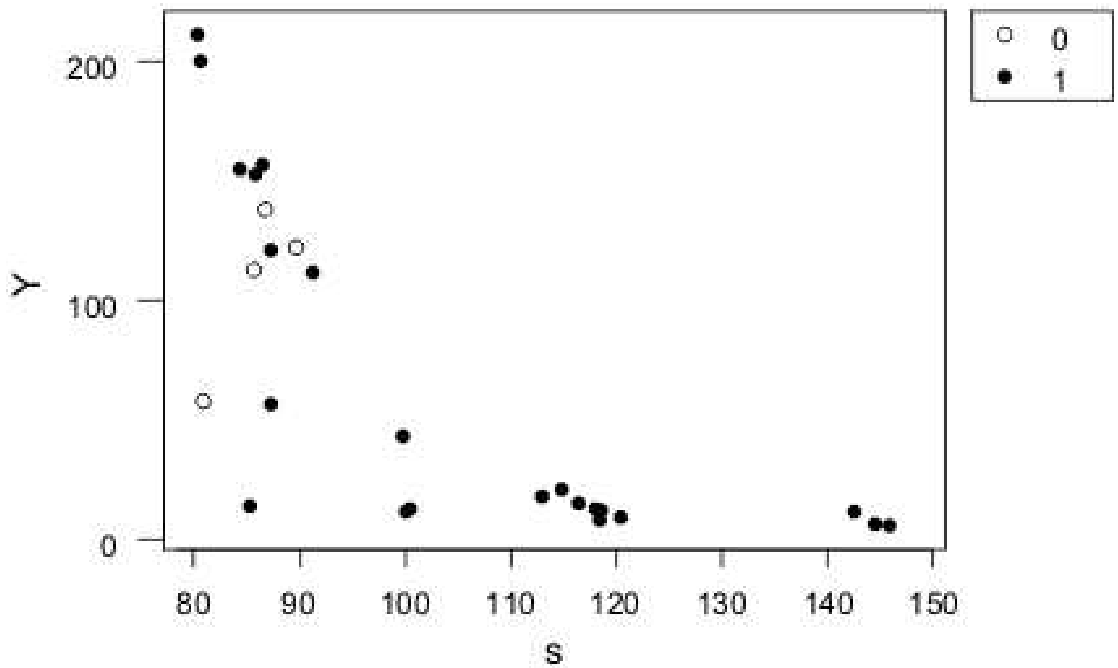


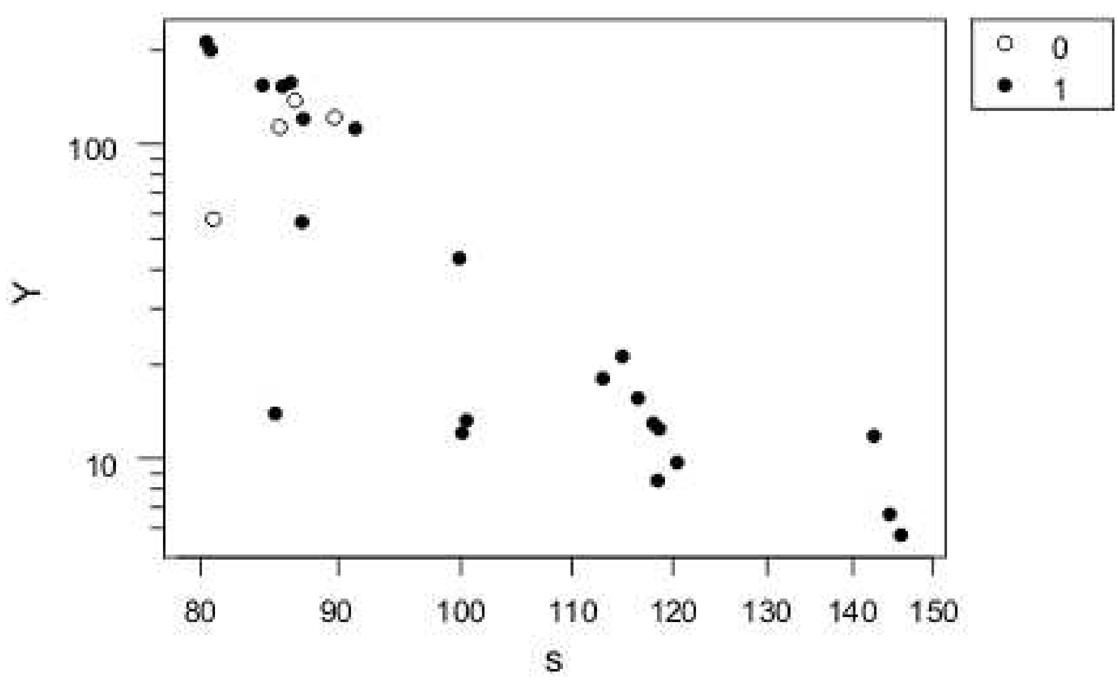


Row i	Pseudo-stress s	k-Cycles Y	Status (1=failed, 0=censored) C
1	80,3	211,629	1
2	80,6	200,027	1
3	80,8	57,923	0
4	84,3	155,000	1
5	85,2	13,949	1
6	85,6	112,968	0
7	85,8	152,680	1
8	86,4	156,725	1
9	86,7	138,114	0
10	87,2	56,723	1
11	87,3	121,075	1
12	89,7	122,372	0
13	91,3	112,002	1
14	99,8	43,331	1
15	100,1	12,076	1
16	100,5	13,181	1
17	113,0	18,067	1
18	114,8	21,300	1
19	116,4	15,616	1
20	118,0	13,030	1
21	118,4	8,489	1
22	118,6	12,434	1
23	120,4	9,750	1
24	142,5	11,865	1
25	144,5	6,705	1
26	145,9	5,733	1

DATA DESCRIPTION:
Low-Cycle Fatigue Life of Nickel-Base Superalloy Specimens (in units of thousands of cycles to failure).

Data from Nelson (1990).


200
100
0
Y
80
90
100
110
120
130
140
150
s
0
1


100
10
Y
80
90
100
110
120
130
140
150
s
0
1

Regression with Life Data: Y versus x

Response Variable: Y

Censoring Information	Count
Uncensored value	22
Right censored value	4

Censoring value: C = 0

Estimation Method: Maximum Likelihood
Distribution: Weibull

Regression Table

Predictor	Coef	Standard Error	Z	P	95,0% Normal CI Lower	Upper
Intercept	31,432	2,008	15,65	0,000	27,496	35,368
x	-5,9600	0,4329	-13,77	0,000	-6,8085	-5,1116
Shape	2,2105	0,3894			1,5651	3,1221

Log-Likelihood = -97,155

Anderson-Darling (adjusted) Goodness-of-Fit

Standardized Residuals = 1,0768; Cox-Snell Residuals = 1,0768

Regression with Life Data: Y versus x

Response Variable: Y

Censoring Information	Count
Uncensored value	22
Right censored value	4

Censoring value: C = 0

Estimation Method: Maximum Likelihood
Distribution: Weibull

Regression Table

Predictor	Coef	Standard Error	Z	P	95,0% Normal CI Lower	Upper
Intercept	217,61	62,13	3,50	0,000	95,83	339,39
x	-85,52	26,55	-3,22	0,001	-137,55	-33,49
x*x	8,483	2,831	3,00	0,003	2,934	14,032
Shape	2,6685	0,4777			1,8789	3,7900

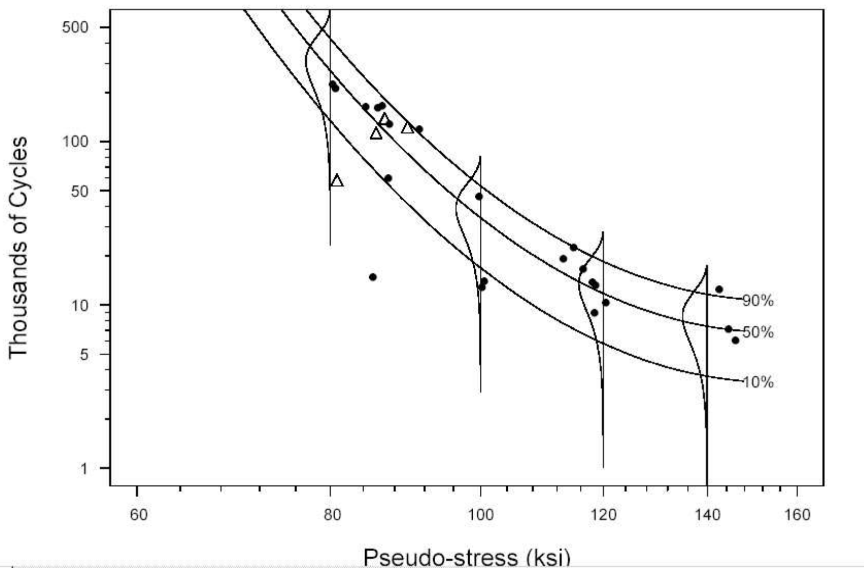
Log-Likelihood = -93,382

Anderson-Darling (adjusted) Goodness-of-Fit

Standardized Residuals = 0,9283; Cox-Snell Residuals = 0,9283

Log-Quadratic Weibull Regression Model with Constant ($\beta = 1/\sigma$) Fit to the Fatigue Data

$$\log[\widehat{t_p}(x)] = \widehat{\mu}(x) + \Phi_{\text{sev}}^{-1}(p)\widehat{\sigma}, \; x = \log(\textbf{pseudo-stress})$$



Regression with Life Data: Y versus x

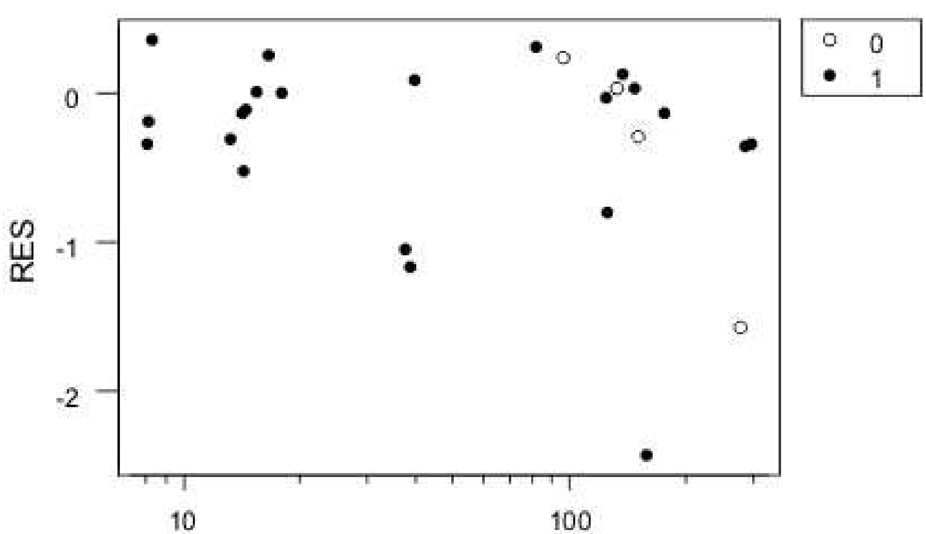
Response Variable: Y

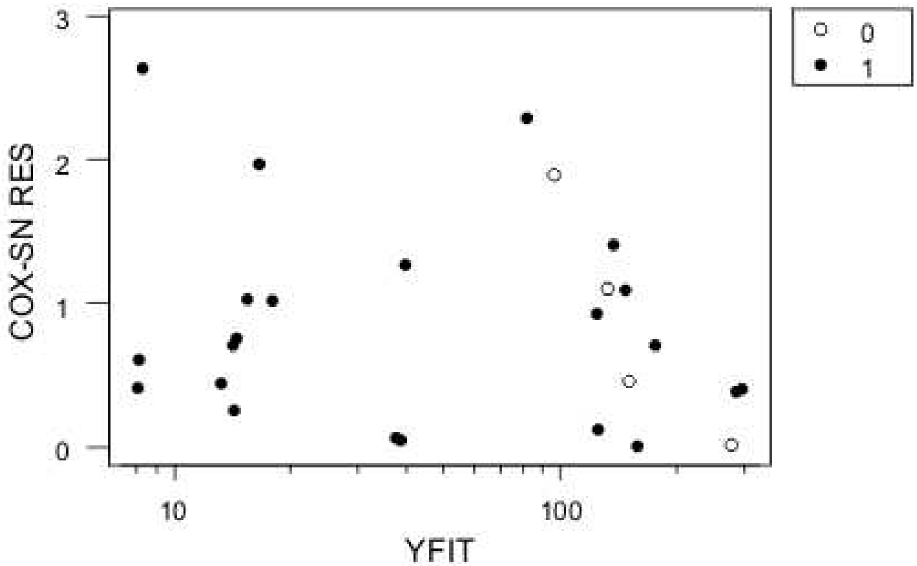
Table of Percentiles

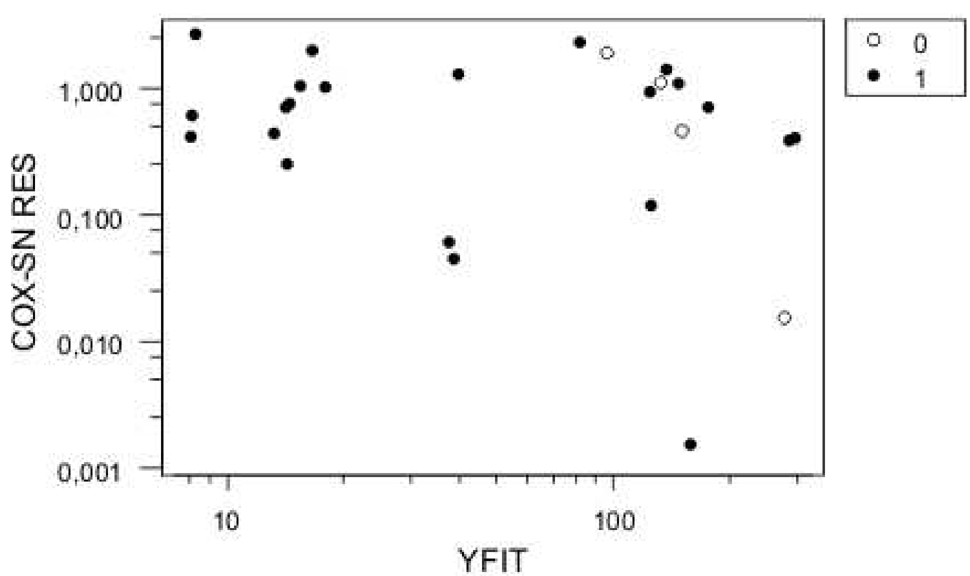
Percent	s	x	Percentile	Standard Error	95,0% Normal CI Lower	95,0% Normal CI Upper
10	80	4,3820	133,3747	34,0579	80,8565	220,0048
10	100	4,6052	16,7928	3,4263	11,2577	25,0494
10	120	4,7875	5,7830	1,2364	3,8034	8,7929
10	140	4,9416	3,6458	0,8760	2,2766	5,8386
50	80	4,3820	270,1879	56,0580	179,9121	405,7621
50	100	4,6052	34,0186	4,3027	26,5494	43,5891
50	120	4,7875	11,7151	1,5950	8,9713	15,2980
50	140	4,9416	7,3856	1,2828	5,2547	10,3807
90	80	4,3820	423,6933	90,4646	278,8097	643,8659
90	100	4,6052	53,3461	6,8162	41,5281	68,5272
90	120	4,7875	18,3709	2,4567	14,1351	23,8760
90	140	4,9416	11,5817	1,9813	8,2824	16,1952

ESTIMERT KOVARIANSMATRISE FOR $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\sigma})$

3860,37	-1649,17	175,82	-0,80
-1649,17	704,70	-75,15	0,33
175,82	-75,15	8,02	-0,03
-0,80	0,33	-0,03	0,23




3
2
1
0
COX-SN RES
10
100
YFIT
0
1


1,000
0,100
0,010
0,001
COX-SN RES
10
100
YFIT
0
1