WEEK 14, 2006
April 6 and 7

TMA4275 LIFETIME ANALYSIS

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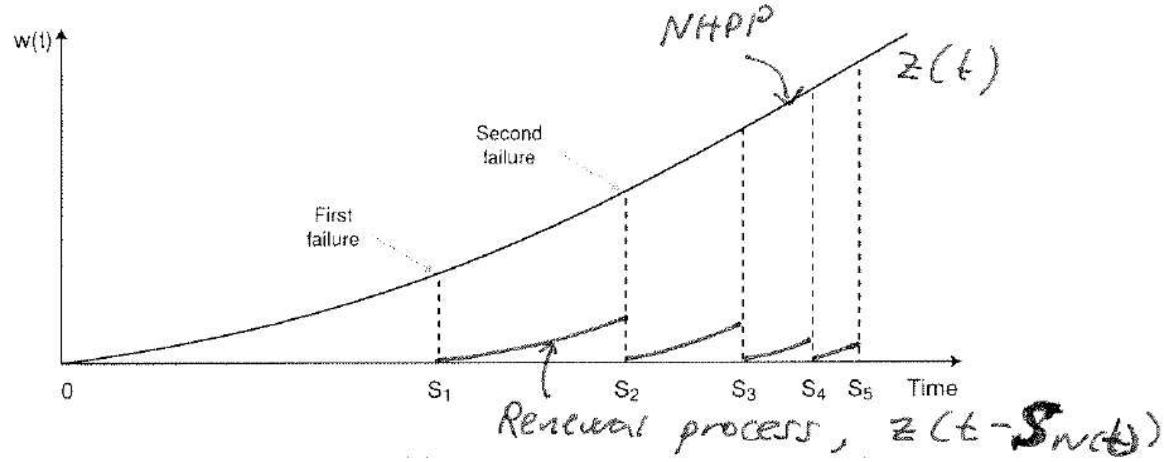
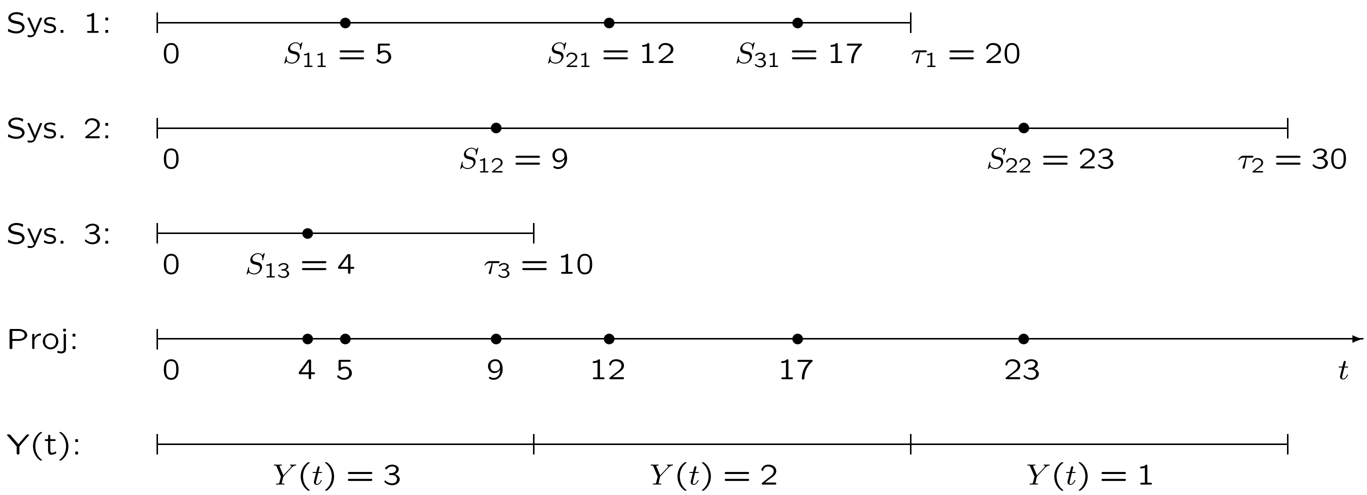
CONDITIONAL ROCOF BY MINIMAL REPAIR (NHPP) AND PERFECT REPAIR (RENEWAL PROCESS)



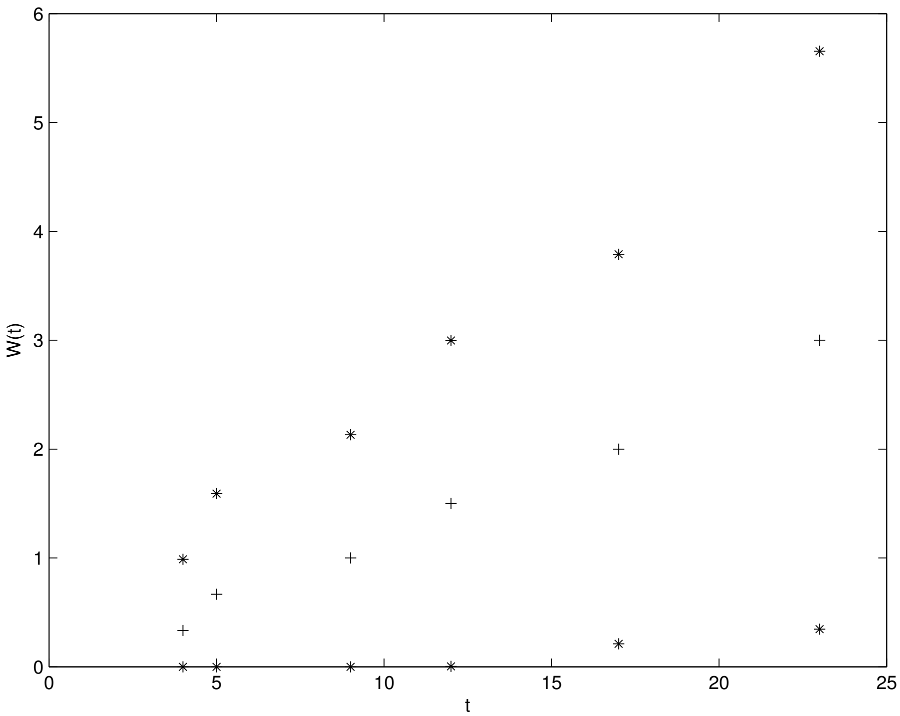
SIMPLE EXAMPLE WITH THREE SYSTEMS



COMPUTATIONS FOR THE NELSON-AALEN ESTIMATOR

t	$1/Y(t)$	$1/Y(t)^2$	$\widehat{W}(t)$	$\widehat{Var\widehat{W}}(t)$	$\widehat{SD\widehat{W}}(t)$
4	1/3	1/9	1/3	1/9	0.3333
5	1/3	1/9	2/3	2/9	0.4714
9	1/3	1/9	1	1/3	0.5774
12	1/2	1/4	3/2	7/12	0.7638
17	1/2	1/4	2	5/6	0.9129
23	1	1	3	11/6	1.3540

ESTIMATED $W(t)$ with 95% confidence limits (Nelson-Aalen)

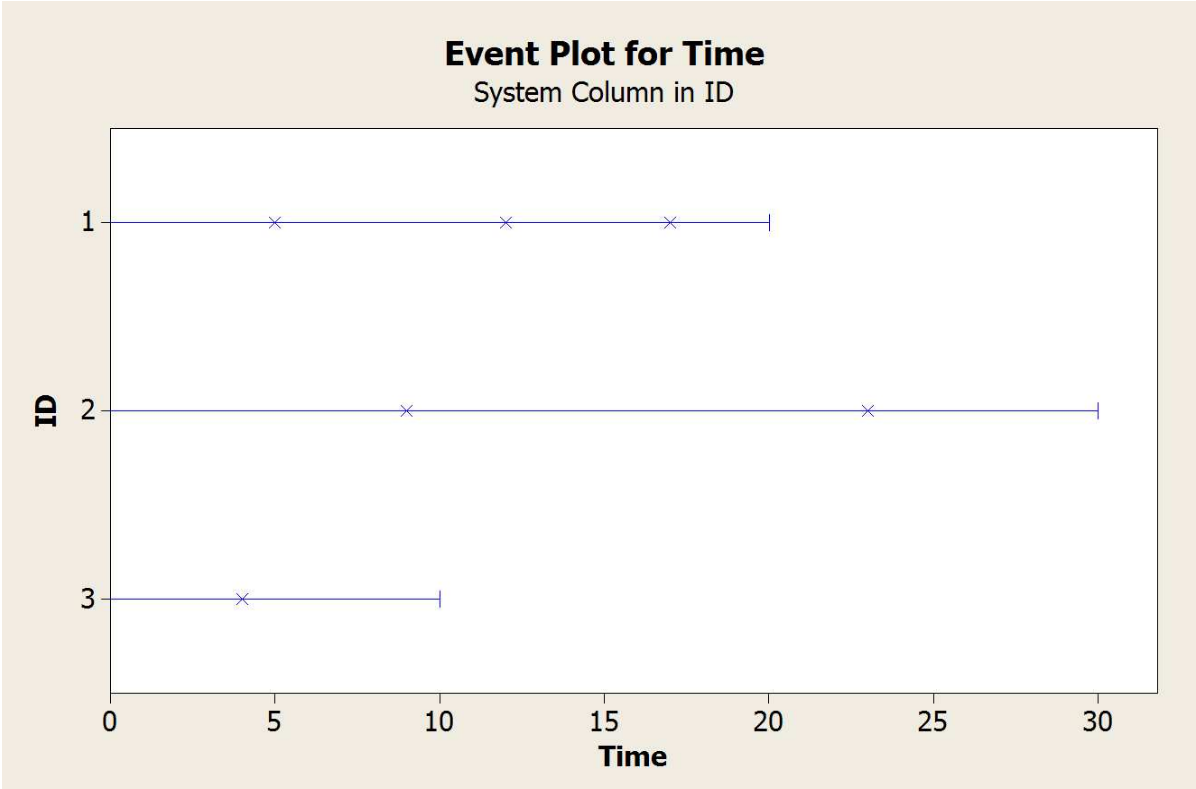


Simple Example With 3 Systems

SimpleNHPP.MTW ***

↓	C1	C2	C3
	ID	Time	
1	1	5	
2	1	12	
3	1	17	
4	1	20	
5	2	9	
6	2	23	
7	2	30	
8	3	4	
9	3	10	
10			

Simple Example With 3 Systems



Simple Example With 3 Systems

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 Results for: SimpleNHPP.MTW
Nonparametric Growth Curve:  Time System:  ID

Nonparametric Estimates

Table of Mean Cumulative Function
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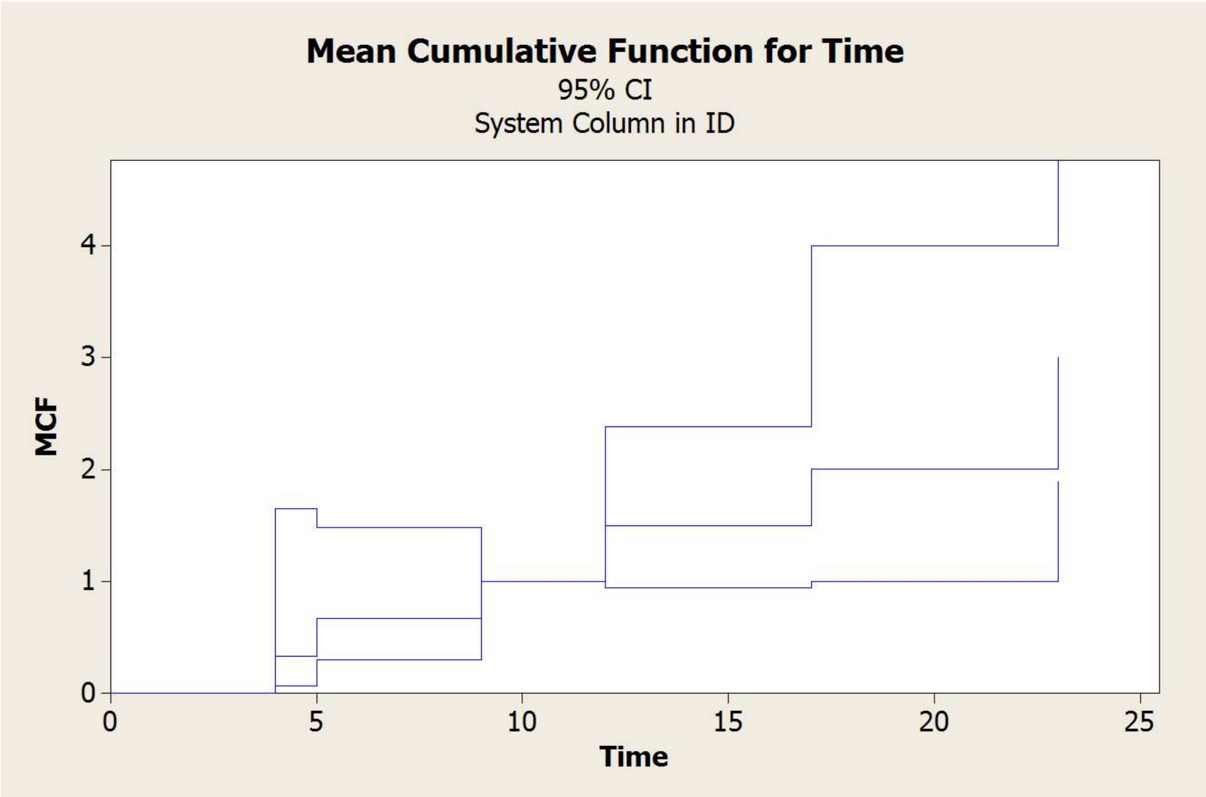
Time	Mean Cumulative Function	Standard Error	95% Normal CI Lower	95% Normal CI Upper	System
4	0,33333	0,272166	0,06728	1,65151	3
5	0,66667	0,272166	0,29951	1,48392	1
9	1,00000	0,000000	1,00000	1,00000	2
12	1,50000	0,353553	0,94506	2,38079	1
17	2,00000	0,707107	1,00020	3,99922	1
23	3,00000	0,707107	1,89013	4,76158	2

Simple Example With 3 Systems



Nelson-Aalen estimator for Cumulative ROCOF $W(t)$

1. Order all failure times as $t_1 < t_2 < \ldots t_n$.

2. Let $d_j(t_i) = \#$ events in system j at t_i.

3. Let $d(t_i) = \sum_{j=1}^{m} d_j(t_i) = \#$ events in all systems at t_i.

4. Let $Y_j(t) = \begin{cases} 1 & \text{if system } j \text{ is under observation at time } t \\ 0 & \text{otherwise} \end{cases}$

5. Let $Y(t) = \sum_{j=1}^{m} Y_j(t) = \#$ systems under observation at time t .

Then

Under general assumptions: $\widehat{W}(t) = \sum_{t_i \le t} \frac{d(t_i)}{Y(t_i)}$.

Assuming NHPP: $\widehat{\text{Var}\,\widehat{W}(t)} = \sum_{t_i \le t} \frac{d(t_i)}{\{Y(t_i)\}^2}$

Under general assumptions (MINITAB): $\widehat{\text{Var}\,\widehat{W}(t)} = \sum_{j=1}^{m} \left\{ \sum_{t_i \le t} \frac{Y_j(t_i)}{Y(t_i)} \left[d_j(t_i) - \frac{d(t_i)}{Y(t_i)} \right] \right\}^2$

Illustration of last formula for Simple NHPP Example (Compare with MINITAB Output):

$$\begin{aligned}
\mathrm{Var}\ \widehat{\widehat{W}}(4) &= \left\{\frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 + \left\{\frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 + \left\{\frac{1}{3}\left[1-\frac{1}{3}\right]\right\}^2 \\
&= \frac{6}{81} = 0.2722^2
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}\ \widehat{\widehat{W}}(5) &= \left\{\frac{1}{3}\left[0-\frac{1}{3}\right] + \frac{1}{3}\left[1-\frac{1}{3}\right]\right\}^2 \\
&+ \left\{\frac{1}{3}\left[0-\frac{1}{3}\right] + \frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 \\
&+ \left\{\frac{1}{3}\left[1-\frac{1}{3}\right] + \frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 \\
&= \frac{6}{81} = 0.2722^2
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}\ \widehat{\widehat{W}}(9) &= \left\{\frac{1}{3}\left[0-\frac{1}{3}\right] + \frac{1}{3}\left[1-\frac{1}{3}\right] + \frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 \\
&+ \left\{\frac{1}{3}\left[0-\frac{1}{3}\right] + \frac{1}{3}\left[0-\frac{1}{3}\right] + \frac{1}{3}\left[1-\frac{1}{3}\right]\right\}^2 \\
&+ \left\{\frac{1}{3}\left[1-\frac{1}{3}\right] + \frac{1}{3}\left[0-\frac{1}{3}\right] + \frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}\ \widehat{\widehat{W}}(12) &= \left\{\frac{1}{3}\left[0-\frac{1}{3}\right] + \frac{1}{3}\left[1-\frac{1}{3}\right] + \frac{1}{3}\left[0-\frac{1}{3}\right] + \frac{1}{2}\left[1-\frac{1}{2}\right]\right\}^2 \\
&+ \left\{\frac{1}{3}\left[0-\frac{1}{3}\right] + \frac{1}{3}\left[0-\frac{1}{3}\right] + \frac{1}{3}\left[1-\frac{1}{3}\right] + \frac{1}{2}\left[0-\frac{1}{2}\right]\right\}^2 \\
&+ \left\{\frac{1}{3}\left[1-\frac{1}{3}\right] + \frac{1}{3}\left[0-\frac{1}{3}\right] + \frac{1}{3}\left[0-\frac{1}{3}\right]\right\}^2 \\
&= \frac{1}{8} = 0.3536^2
\end{aligned}$$

Simple Example With 3 Systems

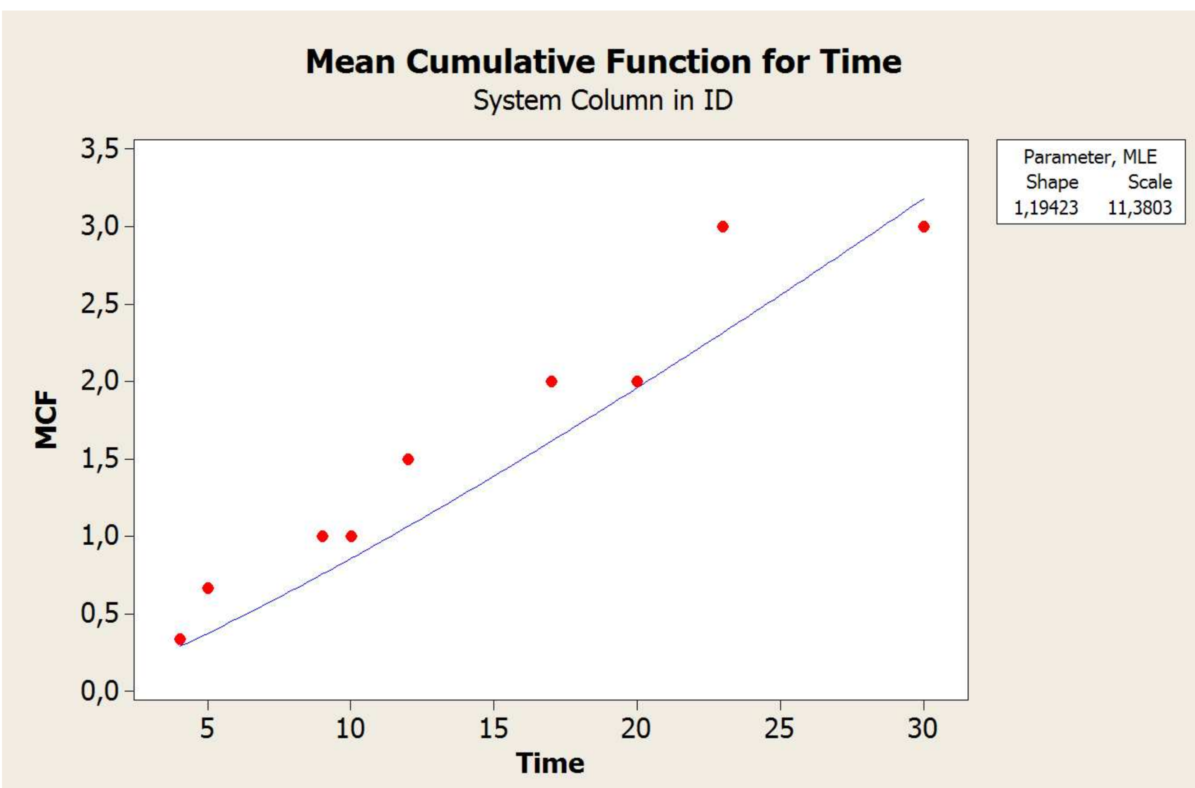
Power Law NHPP Model: $W(t; \alpha, \theta) = (t/\theta)^{\alpha}$

Results for: SimpleNHPP.MTW

Parametric Growth Curve: Time

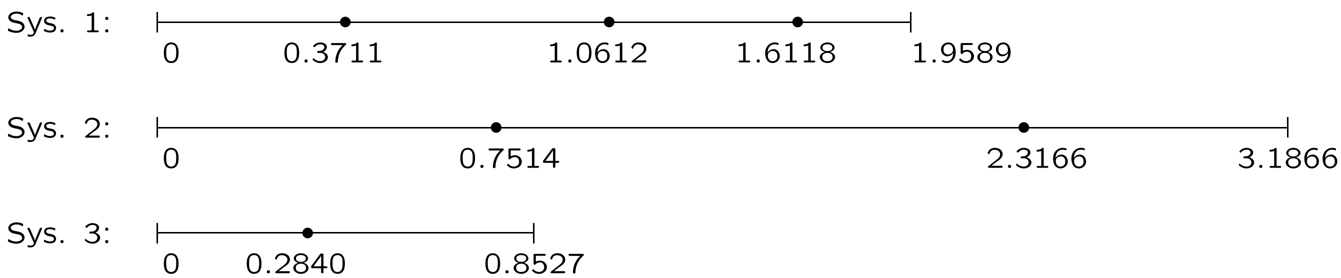
```
System:  ID

Model: Power-Law Process
Estimation Method: Maximum Likelihood


Parameter Estimates

                     Standard    95% Normal CI
Parameter  Estimate     Error     Lower    Upper
Shape       1,19423     0,445  0,323015  2,06545
Scale       11,3803     4,840   1,89335  20,8672


Test for Equal Shape Parameters
Bartlett's Modified Likelihood Ratio Chi-Square

Test Statistic   0,06
P-Value         0,972
DF                  2


Trend Tests

                  MIL-Hdbk-189        Laplace's
                TTT-based  Pooled  TTT-based  Pooled  Anderson-Darling
Test Statistic       9,03    8,89       0,28    0,31              0,28
P-Value             0,599   0,576      0,781   0,756             0,954
DF                     12      12
```


Simple Example With 3 Systems

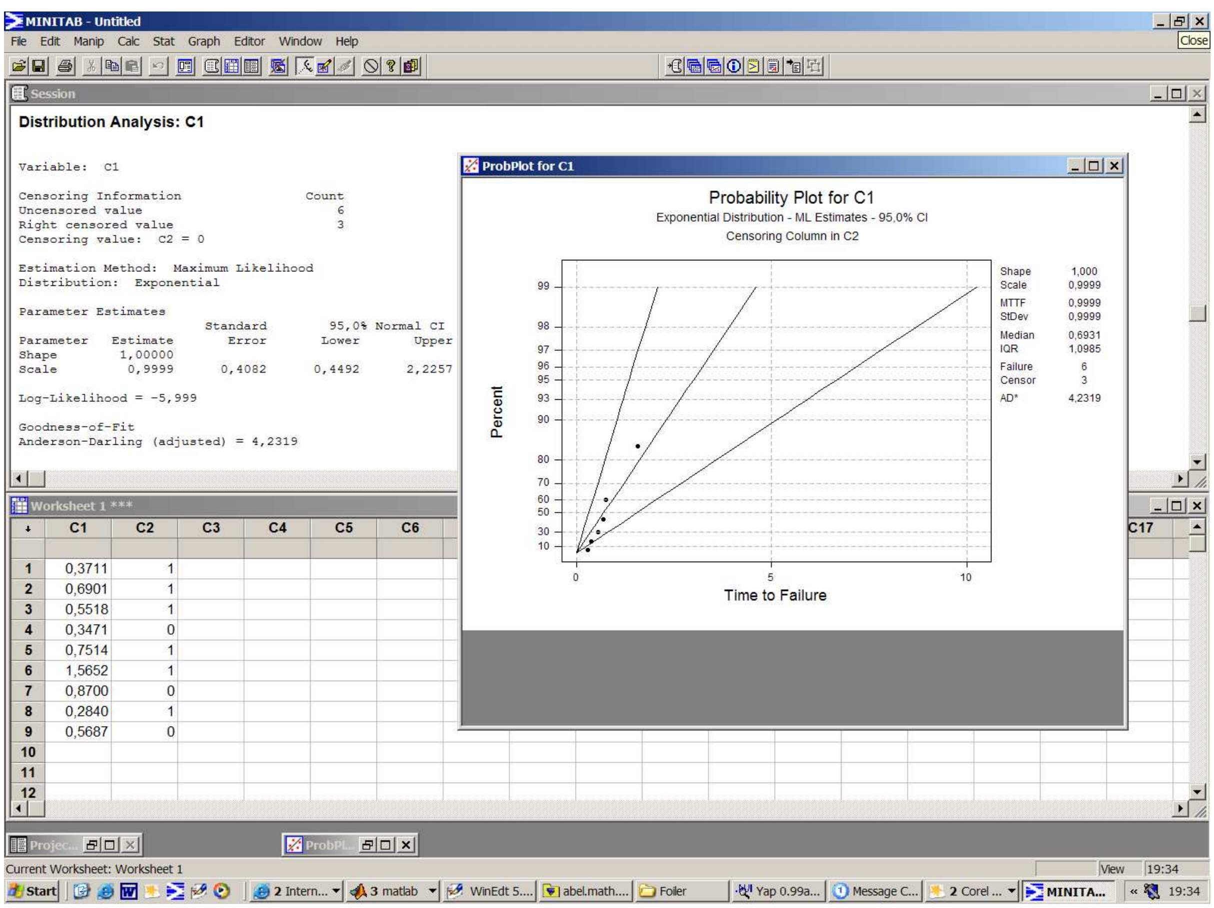


RESIDUAL PROCESS: "SIMPLE EXAMPLE".
Data points (and endpoints on axes) are transformed with the estimated cumulative ROCOF,

$$\widehat{W}(t) = 0.0538 \cdot t^{1.20}$$



Times between events, plus censored times at the end of each axis, are on the next slide anlysed by MINITAB as a set of censored exponential variables.



Valve Seat Replacement Times (Nelson and Doganaksoy 1989)

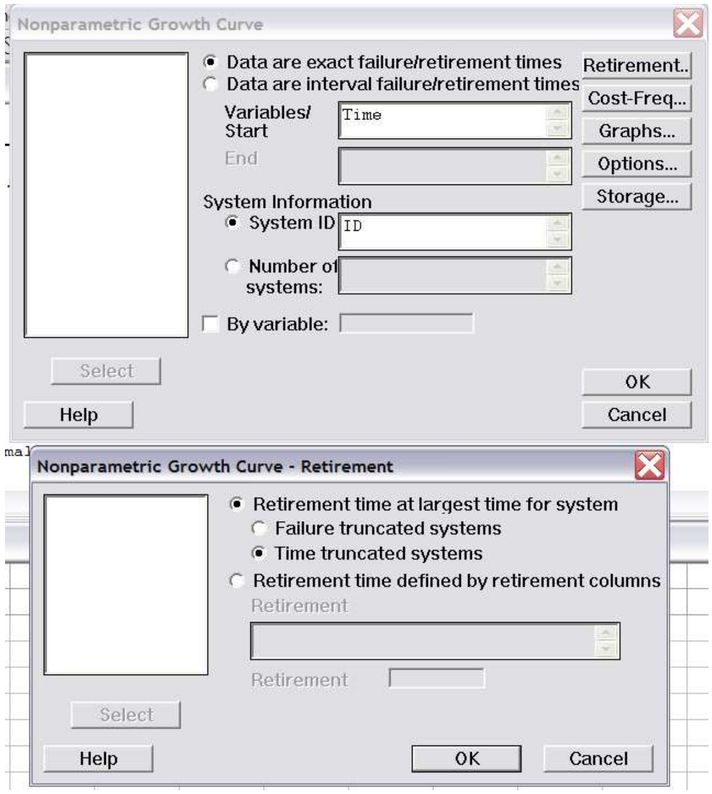
Data collected from valve seats from a fleet of 41 diesel engines (days of operation)

- Each engine has 16 valves
- Does the replacement rate increase with age?
- How many replacement valves will be needed in the future?
- Can valve life in these systems be modeled as a renewal process?

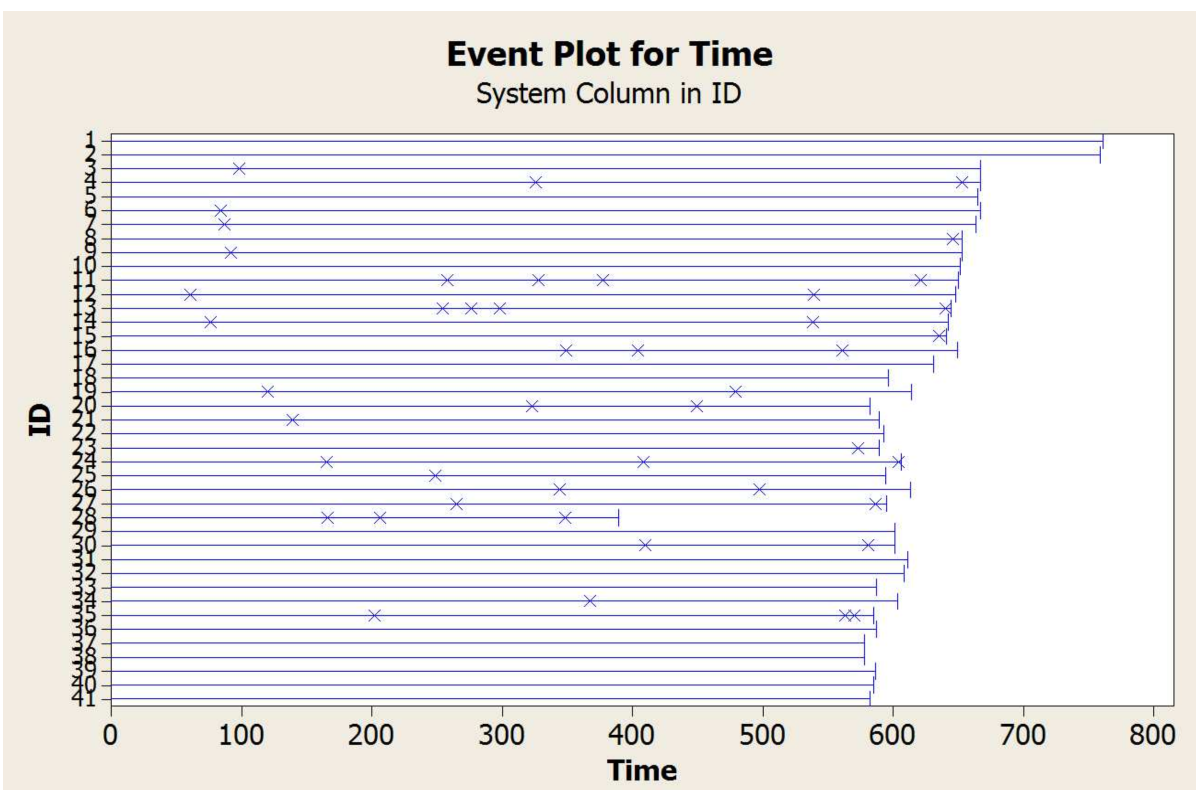
VALVESEAT DATA

TMA4275valveseat.MTW ***

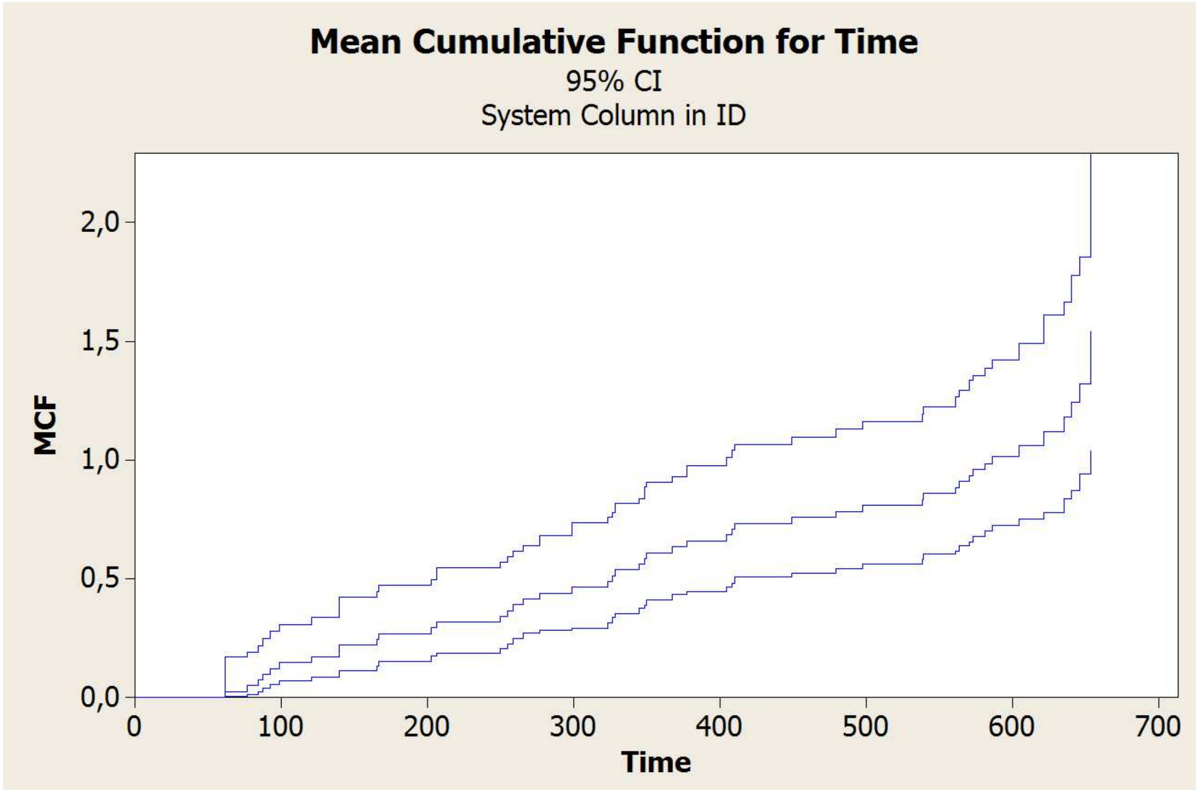
↓	C1	C2	C3	C4	C5	C6	C7	C8
	ID	Time						
1	1	761						
2	2	759						
3	3	98						
4	3	667						
5	4	326						
6	4	653						
7	4	653						
8	4	667						
9	5	665						
10	6	84						
11	6	667						
12	7	87						
13	7	663						
14	8	646						
15	8	653						
16	9	92						
17	9	653						
18	10	651						
19	11	258						
20	11	328						
21	11	377						
22	11	621						
23	11	650						
24	12	61						
25	12	539						
26	12	648						



VALVESEAT DATA



VALVESEAT DATA



VALVESEAT DATA

Nonparametric Growth Curve: Time

System: ID

Nonparametric Estimates

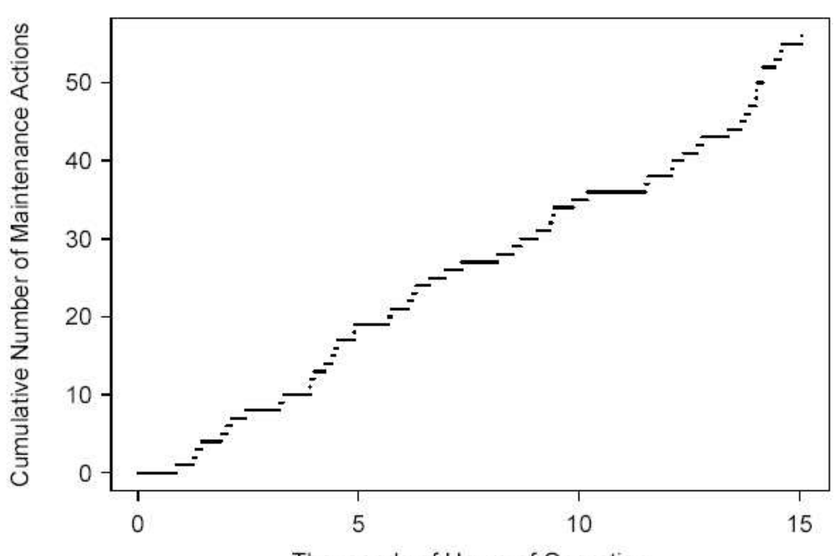
Table of Mean Cumulative Function

Time	Mean Cumulative Function	Standard Error	95% Normal CI Lower	95% Normal CI Upper	System
61	0,02439	0,024091	0,00352	0,16903	12
76	0,04878	0,033641	0,01262	0,18848	14
84	0,07317	0,040670	0,02462	0,21750	6
87	0,09756	0,046340	0,03846	0,24750	7
92	0,12195	0,051105	0,05364	0,27726	9
98	0,14634	0,055199	0,06987	0,30650	3
120	0,17073	0,058764	0,08696	0,33519	19
139	0,19512	0,061891	0,10479	0,36333	21
139	0,21951	0,073270	0,11411	0,42226	21
165	0,24390	0,075417	0,13305	0,44711	24
166	0,26829	0,077317	0,15251	0,47196	28
202	0,29268	0,078988	0,17246	0,49672	35
206	0,31707	0,087527	0,18458	0,54467	28
249	0,34146	0,088680	0,20525	0,56807	25
254	0,36585	0,089656	0,22631	0,59143	13
258	0,39024	0,090461	0,24775	0,61468	11
265	0,41463	0,091101	0,26955	0,63780	27
276	0,43902	0,097858	0,28363	0,67955	13
298	0,46341	0,109607	0,29150	0,73671	13
323	0,48780	0,109740	0,31387	0,75812	20
326	0,51220	0,109740	0,33656	0,77949	4
328	0,53659	0,114907	0,35266	0,81643	11
344	0,56098	0,114654	0,37581	0,83737	26
348	0,58537	0,124250	0,38615	0,88737	28
349	0,60976	0,123782	0,40960	0,90772	16
367	0,63415	0,123194	0,43334	0,92801	34
377	0,65854	0,131842	0,44480	0,97498	11
404	0,68354	0,135939	0,46289	1,00936	16

Times of Unscheduled Maintenance Actions for a USS Grampus Diesel Engine

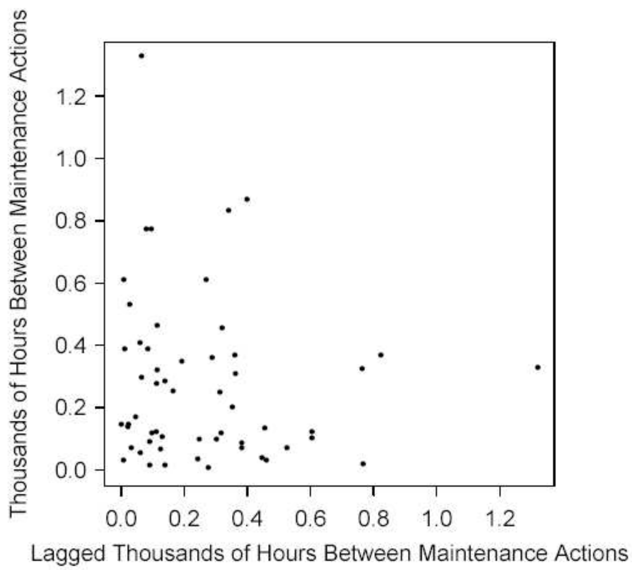
- Unscheduled maintenance actions caused by failure of imminent failure.
- Unscheduled maintenance actions are inconvenient and expensive.
- Data available for 16,000 operating hours.
- Data from Lee (1980).
- Is the system deteriorating (i.e., are failures occurring more rapidly as the system ages)?
- Can the occurrence of unscheduled maintenance actions be modeled by an HPP?

Cumulative Number of Unscheduled Maintenance Actions Versus Operating Hours for a USS Grampus Diesel Engine Lee (1980)



Grampus- data: Plot of (T_i, T_{i+1}) to investigate whether times between failures can be assumed independent. The figure does not indicate a correlation between successive times.

USS Grampus Diesel Engine Plot of Times Between Unscheduled Maintenance Actions Versus Lagged Times Between Unscheduled Maintenance Actions



The Likelihood for the NHPP - Single Unit

- With **interval** recurrence data.

  Suppose that the unit has been observed for a period $(0, t_a]$ and the data are the number of recurrences $d_1, \ldots, d_m$ in the nonoverlapping intervals $(t_0, t_1]$, $(t_1, t_2]$, $\ldots$, $(t_{m-1}, t_m]$ (with $t_0 = 0$, $t_m = t_a$).

$$
\begin{aligned}
L(\boldsymbol{\theta}) &= \Pr\left[N(t_0, t_1) = d_1, \ldots, N(t_{m-1}, t_m) = d_m\right] \\
&= \prod_{j=1}^{m} \Pr\left[N(t_{j-1}, t_j) = d_j\right] \\
&= \prod_{j=1}^{m} \frac{\left[\mu(t_{j-1}, t_j; \boldsymbol{\theta})\right]^{d_j}}{d_j!} \exp\left[-\mu(t_{j-1}, t_j; \boldsymbol{\theta})\right] \\
&= \prod_{j=1}^{m} \frac{\left[\mu(t_{j-1}, t_j; \boldsymbol{\theta})\right]^{d_j}}{d_j!} \times \exp\left[-\mu(t_0, t_a; \boldsymbol{\theta})\right]
\end{aligned}
$$

The Likelihood for the NHPP (Continued)

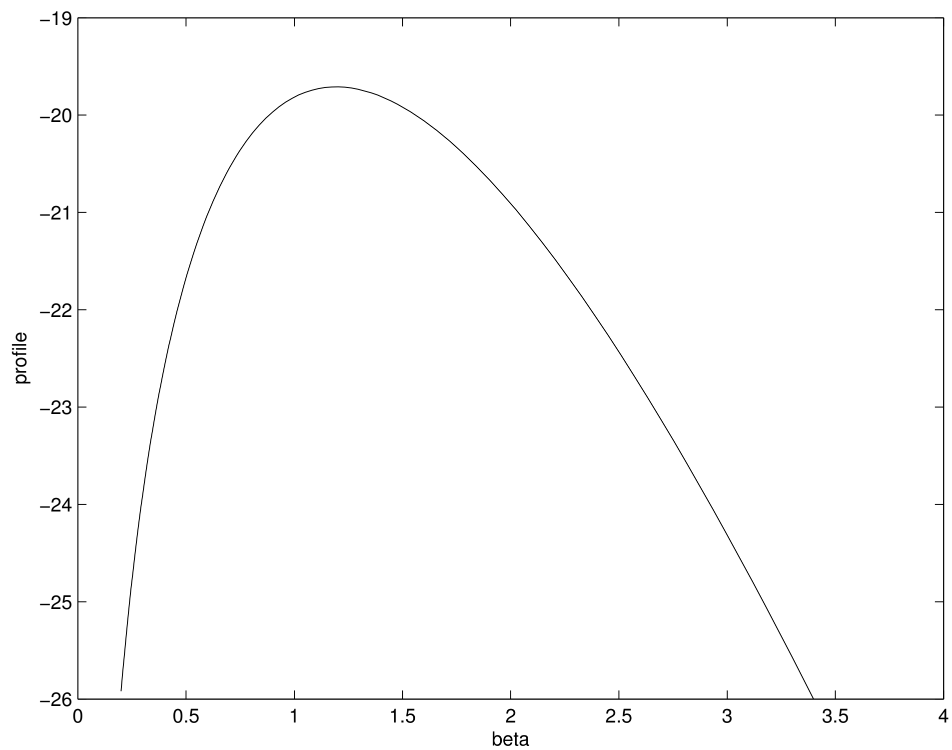
- If the number of intervals m increases and there are **exact** recurrences at $t_1 \le \ldots \le t_r$ (here $r = \sum_{j=1}^{m} d_j$, $t_0 \le t_1$, $t_r \le t_a$), then using a limiting argument it follows that the likelihood in terms of the density approximation is

$$L(\boldsymbol{\theta}) = \prod_{j=1}^{r} \nu(t_j; \boldsymbol{\theta}) \times \exp\left[-\mu(0, t_a; \boldsymbol{\theta})\right]$$

- For simplicity, above we assumed that the intervals are contiguous. Obvious changes to the formula above give the likelihood when there are gaps among the intervals.

- In both cases (the interval data or exact recurrences data) the same methods used in Chapters 7, 8 can be used to obtain the ML estimate $\widehat{\boldsymbol{\theta}}$ and confidence regions for $\boldsymbol{\theta}$ or functions of $\boldsymbol{\theta}$.

PROFILE LIKELIHOOD FOR BETA
("SIMPLE EXAMPLE")
$\widehat{\beta} = 1.20,\ \widehat{\lambda} = 0.0538.$



CONNECTION BETWEEN LAMBDA OG BETA
("SIMPLE EXAMPLE")
$\hat{\beta} = 1.20,\ \hat{\lambda} = 0.0538.$