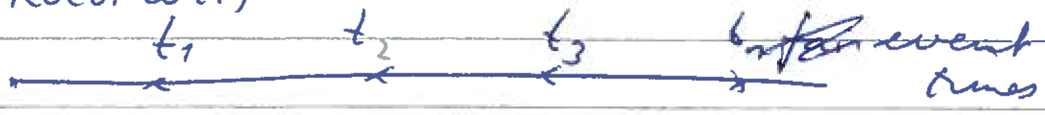
FORELESNING 21
26/3-09

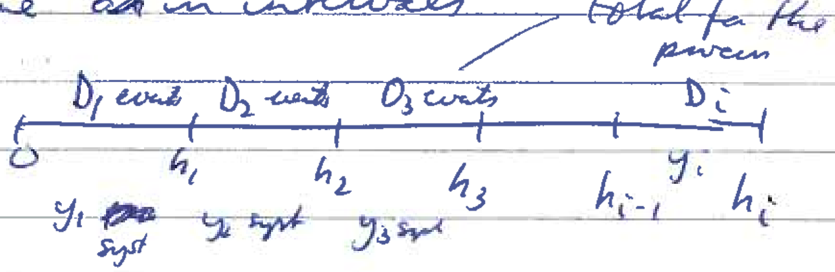
From last time:

Rep. system, ROCOF $w(t)$



~~ROCOF (intensity) $w(t)$~~

(1) If we divide time ~~at~~ in intervals $0 = h_1 < h_2 < \cdots$ then



$$\hat{W}(h_k) = \sum_{i=1}^{k} \frac{D_i}{y_i}$$

(2) Letting $h_1, h_2, \ldots$ more and more dense

$$\Longrightarrow \hat{W}(t) = \sum_{t_i \le t} \frac{d(t_i)}{Y(t_i)}$$

usually 1

Variance of $\hat{W}$:

(1) If the process is NHPP with $w(t)$

then $D_i \sim \text{Poisson}\ \cancel{(y_i(h_i - h_{i-1}))}$

$$\Big(y_i\,\underbrace{(W(h_i) - W(h_{i-1}))}_{\text{one process}}\Big)$$

number of processes

[We use that if $X_1, X_2, \dots, X_n$ are Poisson (λ_i) then $\sum_{i=1}^{n} X_i \sim \text{Poisson}\left(\sum_{i=1}^{n} \lambda_i\right)$]

Then $E(D_i) = y_i\,(W(h_i) - W(h_{i-1}))$

and $\operatorname{Var}(D_i) = E(D_i)$ [property of Poisson-distr]

& $D_1, D_2, \dots$ indep [property of NHPP]

Thus

$$\operatorname{Var} \hat{W}(h_k) = \operatorname{Var}\left(\sum_{i=1}^{k} \frac{D_i}{y_i}\right)$$

$$= \sum_{i=1}^{k} \frac{1}{y_i^2} \operatorname{Var}(D_i)$$

$$= \sum_{i=1}^{k} \frac{1}{y_i^2} \cdot E(D_i)$$

But $E(D_i)$ can be estimated by D_i, so

$$\widehat{\operatorname{Var} \hat{W}(h_k)} = \sum_{i=1}^{k} \frac{D_i}{y_i^2} \qquad \left(\hat{W}(h_k) = \sum_{i=1}^{k} \frac{D_i}{y_i}\right)$$

for an NHPP

- taking limit : $\widehat{\operatorname{Var} \hat{W}(t)} = \sum_{t_i \le t} \frac{d(t_i)}{Y(t_i)^2}$

But - with <u>not</u> NHPP: $\hat{W}(t)$ still valid, but other $\mathrm{Var}\,\hat{W}(t)$ has to be used.

Slides ~~[illegible]~~ 168 - 177 & 185 - 187 (Valve seat data). gives example etc.

MINITAB: Stat > Reliability/Survival
> Repairable System Analysis
> Nonparametric Growth Curve

Parametric estimation in NHPP:

NHPP is characterized by the ROCOF (intensity) $w(t)$.

Popular models:

[Notation: $W(t) = \int_0^t w(u)\,du \equiv E[N(t)]$

$$W(s,t) = \int_s^t w(u)\,du = E[\underbrace{N(s,t)}_{\text{\# events in interval } (s,t)}]$$

]

Power law ~~process~~ NHPP:

$$w(t) = \lambda\beta t^{\beta-1} \quad \text{for } \beta > 0.$$

$\downarrow$ if $\beta < 1$

$\uparrow$ if $\beta > 1$

NHPP$_{\ast}$ if $\beta = 1$ $\ast$

$$W(t) = \int_0^t w(u)\,du = \int_0^t \lambda\beta u^{\beta-1}\,du = \lambda t^{\beta}$$

[Similar to Weibull intensity], $\frac{\alpha}{\theta}\left(\frac{t}{\theta}\right)^{\alpha-1}$

$$W(s,t) = \lambda(t^{\beta} - s^{\beta})$$

Log linear NHPP:

$$w(t) = e^{\alpha+\beta t}$$

$\downarrow \lambda \quad \beta < 0$

$\nearrow \lambda \quad \beta > 0$

$\rightarrow \lambda \quad \beta = 0.$

Hence $W(t) = \int_0^t e^{\alpha+\beta u}\, du$

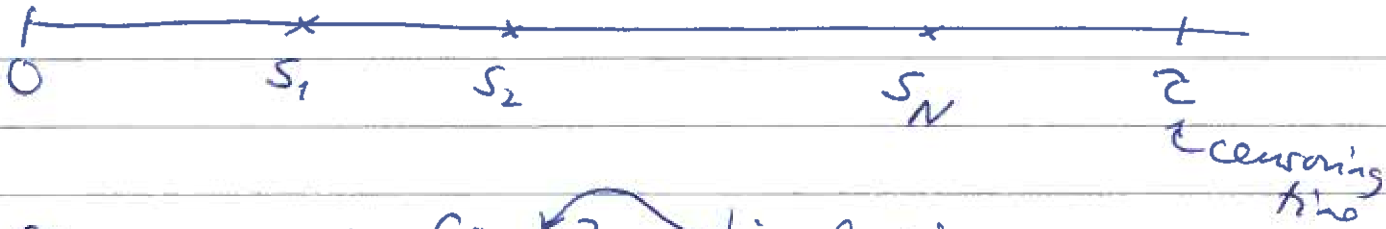
$$= e^{\alpha} \int_0^t e^{\beta u}\, du$$

$$= e^{\alpha} \left[\frac{1}{\beta} e^{\beta u} \right]_0^t = \frac{e^{\alpha}}{\beta}\left(e^{\beta t} - 1\right)$$

$$W(s,t) = \frac{e^{\alpha}}{\beta}\left(e^{\beta t} - e^{\beta s}\right)$$

Likelihood function for NHPP data

Suppose we observe one process; ROCOF $w(t)$



Observ. interval $(0, \tau]$ fixed, given

\# ~~observations~~ events is N (random variable,

$$N \equiv N(\tau) \sim \text{Poisson}(W(\tau)))$$

Times of events: $0 \leq S_1 < S_2 < \cdots < S_N \leq \tau$

Parametric model: Write ROCOF as $w(t; \theta)$

one or more parameters

e.g. $w(t; \lambda, \beta) = \lambda \beta t^{\beta - 1}$

for power law

Define also $W(t; \theta) = \int_0^t w(u; \theta)\, du = E[N(t)]$

$$W(s, t; \theta) = \int_s^t w(u; \theta)\, du = E[N(s,t)]$$

Divide time axis at $h_0 = 0 \le h_1 < h_2 < \cdots < h_r \equiv \tau$

D_i = # events in (h_{i-1}, h_i)

$\sim$ Poisson $\left(W(h_{i-1}, h_i; \theta)\right)$

<u>Observations</u> are $D_1, D_2, \cdots, D_r$
which are independent (property of NHPP)
and Poisson-distributed

Hence likelihood is

the observed ones

$$L(\theta) = P(D_1 = d_1, D_2 = d_2, \cdots, D_r = d_r)$$

$$= \prod_{i=1}^{r} P(D_i = d_i) = \prod_{i=1}^{r} \frac{W(h_{i-1}, h_i; \theta)^{d_i}}{d_i!} e^{-W(h_{i-1}, h_i; \theta)}$$

$$= \left\{ \prod_{i=1}^{r} \frac{W(h_{i-1}, h_i; \theta)^{d_i}}{d_i!} \right\} \cdot e^{-\underbrace{\sum_{i=1}^{r} W(h_{i-1}, h_i; \theta)}}$$

But $$\sum_{i=1}^{r} W(h_{i-1}, h_i; \theta) = \sum_{i=1}^{r} \int_{h_{i-1}}^{h_i} w(u;\theta)\, du$$

$$= \int_0^{\tau} w(u;\theta)\, du = W(\tau; \theta)$$

so

$$L(\theta) = \left\{ \prod_{i=1}^{r} \frac{W(h_{i-1}, h_i; \theta)^{d_i}}{d_i!} \right\} e^{-W(\tau;\theta)}$$

If data are given in groups like this, then we use this to find MLE.

If times are given exactly as $s_1, s_2, \ldots s_N$, then we let the grid of h_i be more and more dense and get in the limit 0 or 1 event in each interval (h_{i-1}, h_i).

Now when $d_i = 0$ is the contribution to the product

$$\frac{W(h_{i-1}, h_i; \theta)^0}{0!} = 1$$

so we can ignore it in $L(\theta)$.

When $d_i = 1$ is the contribution

$$W(h_{i-1}, h_i; \theta) = \int_{h_{i-1}}^{h_i} w(u; \theta) du \approx$$

$$\approx w(s_k)(h_i - h_{i-1})$$

the single s_k in the interval

and since the $h_i - h_{i-1}$ are given by us, we let the contribution to the likelihood be $w(s_k; \theta)$

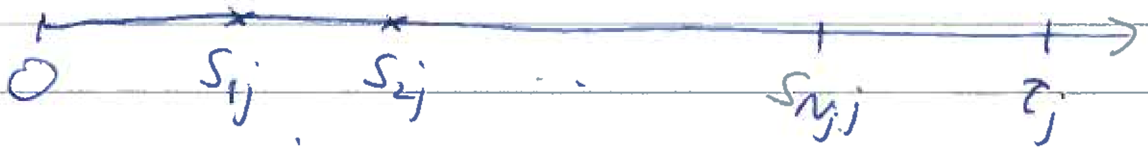
Hence

$$(*) \quad L(\theta) = \underbrace{\prod_{k=1}^{N} w(s_i; \theta)}_{\text{Similar to the } \prod_{i=1}^{n} f(t_i;\theta) \text{ when we have } n \text{ i.i.d. observed failures } t_1, t_2, \dots, t_n.} e^{-W(\tau;\theta)}$$

See also slides 191 – 192 (taken from another course)

Note (*) is for a single system.

If we have data for several systems: with the same $w(t;\theta)$:



$j = 1, 2, \dots, m$

the likelihood is the product:

$$L(\theta) = \prod_{j=1}^{m} \prod_{i=1}^{N_j} w(s_{ij};\theta)\, e^{-W(\tau_j;\theta)}$$

or log-likelihood is

$$\ell(\theta) = \sum_{j=1}^{m} \left[\sum_{i=1}^{N_j} \ln w(s_{ij};\theta) - W(\tau_j;\theta) \right]$$

Example:

$$w(t;\lambda,\beta) = \lambda \beta t^{\beta-1}$$
$$W(t;\lambda,\beta) = \lambda t^{\beta}$$

$$\ln w(t;\lambda,\beta) = \ln\lambda + \ln\beta + (\beta-1)\ln t$$

$$\ell(\lambda,\beta) = \sum_{j=1}^{m} \left[\sum_{i=1}^{N_j} \left(\ln\lambda + \ln\beta + (\beta-1)\ln s_{ij} \right) - \lambda \tau_j^{\beta} \right]$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{N_j} \ln \lambda + \sum_{j=1}^{m} \sum_{i=1}^{N_j} \ln \beta$$

$$+ \sum_{j=1}^{m} \sum_{i=1}^{N_j} (\beta - 1) \ln s_{ij}$$

$$- \sum_{j=1}^{m} \lambda \tau_j{}^{\beta}$$

$$= N \ln \lambda + N \ln \beta + (\beta - 1) S - \lambda \sum_{j=1}^{m} \tau_j{}^{\beta}$$

put $N = \sum_{j=1}^{m} N_j$ where $S = \sum_{j=1}^{m} \sum_{i=1}^{N_j} \ln s_{ij}$

tot. numb. obs.

MLE:

$$\boxed{\begin{aligned} \frac{\partial l}{\partial \lambda} &= \frac{N}{\lambda} - \sum_{j=1}^{m} \tau_j{}^{\beta} = 0 \qquad (1) \\ \frac{\partial l}{\partial \beta} &= \frac{N}{\beta} + S - \lambda \sum_{j=1}^{m} (\tau_j{}^{\beta}) \ln \tau_j = 0 \qquad (2) \end{aligned}}$$

(1) gives $\lambda = \dfrac{N}{\sum_{j=1}^{m} \tau_j{}^{\beta}}$

Put into (2):

$$\frac{N}{\beta} + S - \frac{N \sum_{j=1}^{m} \tau_j^{\beta} \ln \tau_j}{\sum_{j=1}^{m} \tau_j^{\beta}} = 0$$

Can solve this numerically for β to get $\hat{\beta}$
and then put $\hat{\lambda} = N / \sum_{j=1}^{m} \tau_j^{\hat{\beta}}$

Special cases:

If all $\tau_j \equiv \tau$ are equal:
[e.g. if $m = 1$!]

$$\frac{N}{\beta} + S - \frac{N \cancel{m} \cancel{\tau^{\beta}} \ln \tau}{\cancel{m \tau^{\beta}}} = 0$$

$\cancel{\beta}$

$$\frac{N}{\beta} + S - N \ln \tau = 0$$

$$\frac{N}{\beta} = N \ln \tau - S$$

$$\beta = \frac{N}{N \ln \tau - S}$$

So: in this case $\hat{\beta} = \dfrac{N}{N \ln \tau - S}$

$\cancel{\hat{\lambda} = \dfrac{N}{\sum_{j=1}^{m} \tau_j^{\hat{\beta}}}}$

With profile likelihood:

If β is known, then we found

$$\hat{\lambda}(\beta) = \frac{N}{\sum_{j=1}^{m} \tau_j^{\beta}}$$

Profile likelihood of β is therefore

$$\tilde{\ell}(\beta) = \ell(\hat{\lambda}(\beta), \beta) =$$

$$N \ln \hat{\lambda}(\beta) + N \ln \beta + (\beta - 1) S$$

$$- \hat{\lambda}(\beta) \cdot \sum_{j=1}^{m} \tau_j^{\beta}$$

$$= N \ln N - N \ln\left(\sum_{j=1}^{m} \tau_j^{\beta}\right)$$

$$+ N \ln \beta + (\beta - 1) S$$

$$- N$$

Then (1) Maximize $\tilde{\ell}(\beta)$ to find $\hat{\beta}$

(2) Put $\hat{\lambda} = \hat{\lambda}(\hat{\beta})$.

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In simple example:

$$\tilde{\ell}(\beta) = 6 \ln 6 - 6 \ln (20^\beta + 30^\beta + 10^\beta)$$

$$+ 6 \ln \beta + (\beta - 1) \underbrace{13.6466}_{S} - 6$$

~~Fig~~ (Slide 193)

Read off $\hat{\beta} \approx 1.20$

($= 1.19423$ MINITAB)

$$\hat{\lambda}(\hat{\beta}) = \frac{6}{20^{\hat{\beta}} + 30^{\hat{\beta}} + 10^{\hat{\beta}}}$$

$\Rightarrow \hat{\lambda} = 0.0538$ $\quad (\hat{\beta} = 1.20)$

$\phantom{\Rightarrow \hat{\lambda} =}$ 0.0548 $\quad (\hat{\beta} = 1.19423)$

Confidence interval for β using likelihood-theory:

$$W(\beta) = 2\Big(\underbrace{\ell(\hat{\lambda}(\hat{\beta}), \hat{\beta})}_{= \ell(\hat{\lambda}, \hat{\beta})} - \ell(\hat{\lambda}(\beta), \beta)\Big)$$

$$\approx \chi^2_1$$

$\Rightarrow$ (as before) that a 95% (approx) conf. interval for β is obtained by cutting off ~~loglik~~ profile log-likelihood at maximum value $-$ 1.92, i.e.

$$-19.71 - 1.92 = -21.63$$

Read off from slide 193:

Approx 95%: (0.50, 2.25)

Observed information etc:

Recall

$$\ell(\lambda,\beta) = N\ln\lambda + N\ln\beta + (\beta-1)S' - \lambda\sum_{j=1}^{m}\tau_j^{\beta}$$

where $N = \sum_{j=1}^{m} N_j$ $\quad S' = \sum_{j=1}^{m}\sum_{i=1}^{N_j} \ln S_{ij}$

$$\frac{\partial \ell}{\partial \lambda} = \frac{N}{\lambda} - \sum_{j=1}^{m} \tau_j^{\beta}$$

$$\frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{N}{\lambda^2}$$

$$\frac{\partial^2 \ell}{\partial \lambda \partial \beta} = -\sum_{j=1}^{m} (\ln \tau_j)\,\tau_j^{\beta}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{N}{\beta} + S' - \lambda\sum_{j=1}^{m} (\ln\tau_j)\,\tau_j^{\beta}$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = -\frac{N}{\beta^2} - \lambda\sum_{j=1}^{m} (\ln\tau_j)^2\,\tau_j^{\beta}$$

~~Informasjonsmatrise~~: Observed Information Matrix:

$$\begin{bmatrix} -\frac{\partial^2 l}{\partial \lambda^2} & -\frac{\partial^2 l}{\partial \lambda \partial \beta} \\ -\frac{\partial^2 l}{\partial \lambda \partial \beta} & -\frac{\partial^2 l}{\partial \beta^2} \end{bmatrix}_{\substack{\lambda = \hat{\lambda} \\ \beta = \hat{\beta}}}$$

$$\begin{bmatrix} \frac{N}{\hat{\lambda}^2} & \underline{\sum \tau_j^{\hat{\beta}} (\ln \tau_j)} \\ \underline{\sum \tau_j^{\hat{\beta}} (\ln \tau_j)} & \frac{N}{\hat{\beta}^2} + \hat{\lambda} \left(\underline{\sum \tau_j^{\hat{\beta}} (\ln \tau_j)^2} \right) \end{bmatrix}$$

~~I eksempel~~: In simple example:

$$\begin{bmatrix} \frac{6}{0.0538^2} = 2072.9 & 30^{\hat{\beta}} \ln 30 + 20^{\hat{\beta}} \ln 20 + 10^{\hat{\beta}} \ln 10 = 347.03 \\ 347.03 & \frac{6}{1.2^2} + 0.0538 \cdot 1096 = 63.1315 \end{bmatrix}$$

$$\text{Inverte} \quad \begin{array}{c|cc} & \lambda & \beta \\ \lambda & 0.0060 & -0.0332 \\ \beta & & 0.1984 \end{array}$$

so that

$$\widehat{\mathrm{Var}\hat{\beta}} = 0.1984$$

$$\widehat{\mathrm{SD}\,\hat{\beta}} = \sqrt{0.1984} = 0.4454$$

Standard interval 95%:

$$\underset{1.2}{\hat{\beta}} \pm 1.96 \cdot 0.4454$$

$\Rightarrow$ $[0.327, 2.073]$ This is the interval MINITAB gives here

but MINITAB should have used the "standard interval for positive parameters", i.e.

$$1.2 \cdot e^{\pm 1.96 \cdot \frac{0.4454}{1.2}}$$

i.e. $[0.58, 2.48]$

(which is also closer to the likelihood interval).

$$\widehat{\operatorname{Var}\hat{\lambda}} = 0.0060$$

$$\Rightarrow \widehat{SD}(\hat{\lambda}) = 0.0775$$

MINITAB uses $\cancel{\theta}$ ~~other~~ θ with $\cancel{\alpha = \frac{1}{\lambda}}$ $W(x) = \left(\frac{x}{\theta}\right)^{\alpha}$

Standard 95%: $\hat{\lambda} \pm 1.96 \cdot 0.0775$

0.0548

$(0,\ 0.2067)$

negative

while standard for positive:

$$\hat{\lambda} e^{\pm 1.96 \frac{\widehat{SD}(\hat{\lambda})}{\hat{\lambda}}}$$

$$0.0548 \cdot e^{\pm 1.96 \cdot \frac{0.0775}{0.0548}}$$

$(0.0034,\ 0.8761)$

[Note:

$$\widehat{\rho(\hat{\lambda}, \hat{\beta})} = \frac{\widehat{\operatorname{Cov}(\hat{\lambda}, \hat{\beta})}}{\sqrt{\widehat{\operatorname{Var}\hat{\lambda}} \cdot \widehat{\operatorname{Var}\hat{\beta}}}} = \frac{-0.0332}{\sqrt{0.0060 \cdot 0.1984}}$$

Correlation coefficient

$$= -0.9623$$

(very strong correlation).

This is due to a functional relationship:

$$\frac{N}{\hat{\lambda}} = \sum \tau_j^{\hat{\beta}}$$

$$\hat{\lambda} = \frac{N}{\sum \tau_i^{\hat{\beta}}} = \frac{6}{30^{\hat{\beta}} + 20^{\hat{\beta}} + 10^{\hat{\beta}}}$$

See SLIDE 194

[CAN DO CORRESPONDING LIKELIHOOD ANALYSIS FOR LOG-LINEAR NHPP]

MINITAB VS. "MINE" POWER LAW ESTIMATION:

MINITAB: Power law: $\lambda(t) = \frac{\text{Shape}}{\text{Scale}} \left(\frac{t}{\text{Scale}} \right)^{\text{Shape}-1}$

MINE: power law: $\lambda(t) = \lambda \beta t^{\beta - 1}$

$\Rightarrow$ shape $= \beta$ (OK)

$$\lambda = \frac{\cancel{\text{Shape}}\ 1}{\text{scale}^{\text{shape}}} = \frac{\cancel{1.9423}\ 1}{11.3803^{1.9423}} = 0.0548$$

From MINITAB

Example: Grampus data SLIDES 195-196.

Single system:

Power law, $w(t) = \lambda \beta t^{\beta-1}$, gives

$$\hat{\beta} = 1.22, \quad \hat{\lambda} = \frac{1}{0.553^{1.22}} = 2.06$$

Loglinear: $w(t) = e^{\alpha + \beta t}$

$$\hat{\alpha} = 1.01, \quad \hat{\beta} = 0.0377.$$

Slide 195 shows:

$W(t)$: Nelson-Aalen

Power law: $\hat{W}(t) = \hat{\lambda} t^{\hat{\beta}} = 2.06 \cdot t^{1.22}$

Log-linear: $\hat{W}(t) = \frac{e^{\hat{\alpha}}}{\hat{\beta}} (e^{\hat{\beta} t} - 1)$