TMA4285 Time series models
Project 1, Problem 2
Autumn 2015

Consider the zero-mean MA(2) model

$$Z_t = (1 - \theta_1 B - \theta_2 B^2)a_t,$$

where $\{a_t\}_{t=-\infty}^{\infty}$ is Gaussian distributed white noise with zero mean and unit variance.

We will consider 1- and 2-step ahead forecasting for this model for the following two sets of parameter values.

(i) $\theta_1 = 1.7$ and $\theta_2 = -0.8$

(ii) $\theta_1 = -3$ and $\theta_2 = -1$.

a) Compute the autocovariance function of the process for each set of parameter values.

b) Why is the model invertible for parameter set (i) and not (ii)? Write an R-function that computes the AR(∞) representation of an arbitrary MA(2) model and try to apply this to compute the first 20 coefficients $\pi_1, \pi_2, \ldots, \pi_{20}$ for each of the two sets of parameters. Explain what happens in each of the two cases.

b) For the parameter values in set (i), use formulas in section 5.4 in Wei to compute the first 20 weights for the l-step ahead forecasts $\hat{Z}_t(l)$ of Z_{t+l} for lead times $l = 1$ and $l = 2$, based on all observations up to time t, $Z_t, Z_{t-1}, \ldots$. Briefly comment on what happens also for lead times $l \geq 3$ (see eq. 5.4.5). Also compute the forecast error variance for lead times $l = 1, 2, 3$.

c) Suppose that only $Z_t, Z_{t-1}, \ldots, Z_{t-n+1}$ are known instead of the whole history of the process prior to and included Z_t. The optimal forecast of Z_{t+l} is then given by the conditional expectation of $E(Z_{t+l}|Z_t, Z_{t-1}, \ldots, Z_{t-n+1})$, conditioning on the n known observations only. Since $Z_{t+2}, Z_{t+1}, Z_t, Z_{t-1}, \ldots, Z_{t-n+1}$ is jointly multivariate normal, the conditional expectation of both Z_{t+2} and Z_{t+1} depends linearly on $Z_t, Z_{t-1}, \ldots, Z_{t-n+1}$[1]. Use the general formulas in

[1] `https://en.wikipedia.org/wiki/Multivariate_normal_distribution#Conditional_distributions`

wikipedia for conditional expectations of the multivariate normal to numerically compute the weights involved in the 1-step and 2-step ahead forecasts for parameter set (i) for $n = 1, 2, 4, 8, 16$. Examine the difference between the weights computed using this method and the method used in question b). Verify that the difference is small at least for large n.

d) It follows that forecast error variance as defined in Wei, $\mathrm{Var}(Z_{t+l} - \hat{Z}_t(l))$, for a forecast based on only n observations, equals the conditional variance $\mathrm{Var}(Z_{t+l}|Z_t, Z_{t-1}, \ldots, Z_{t-n+1})$. For the multinormal distribution this variance (aswell as the conditional covariance matrix of Z_{t+1}, Z_{t+2}) is independent of $Z_t, Z_{t-1}, \ldots, Z_{t-n+1}$ and given general formulas (see wikipedia). Using these formulas, compute the forecast error variance for lead times $l = 1, 2$ and examine how these change as you increase the number of observed values n. Compare the error variances to the corresponding values in question c).

e) Apply the same method as in question c) to compute the forecast weights for the other set of parameter values (ii) for the same values of n. Wei, on p. 25, states that "in forecasting, a non-invertible process is meaningless" and, on p. 99, that "a meaningful forecast can be derived only for an invertible process". Discuss, based on your results, if these statements are true or not.

Some useful R-functions you may need are `outer` (for easily constructing the necessary pentadiagonal covariance matrix involving the autocovariances computed in question a) and `solve` (for inverting covariance matrices). Also note how certain submatrices needed in the computation of conditional expectations are easily obtained using combinations of sequences such as `1:n` or `(n+1):(n+2)` or similar as indices.

Present your results in a suitable way graphically or using numerical tables.