TMA4285 Time series models
Solution to exercise 3, autumn 2014

Problem 1.

a) The equation is

$$z_t - 1.1z_{t-1} + 0.3z_{t-2} = 0 \quad \text{for} \quad t = 2, 3, \ldots.$$

To find the roots we need to solve

$$C(B) = 1 - 1.1B + 0.3B^2 = 0.$$

This gives the roots

$$B = \frac{1.1 \pm \sqrt{1.1^2 - 4 \cdot 0.3}}{2 \cdot 0.3} = \frac{1.1 \pm 0.1}{0.6} \Rightarrow B = \frac{1}{0.6} \text{ or } B = 2.$$

Thereby $C(R^{-1}) = 0$ has the roots $R_0 = 0.5$ and $R_1 = 0.6$. The two roots differ, and thereby the general solution is

$$z_t = b_0 \cdot 0.5^t + b_1 \cdot 0.6^t \quad \text{for } t = 0, 1, \ldots.$$

b) The equation is

$$z_t - z_{t-1} + z_{t-2} - z_{t-3} = 0 \quad \text{for} \quad t = 3, 4, \ldots.$$

To find the roots we need to solve

$$C(B) = 1 - B + B^2 - B^3 = (1 - B)(1 + B^2) = 0.$$

This gives the roots $B = 1$, $B = i = \sqrt{-1}$ and $B = -i$, and thereby $C(R^{-1}) = 0$ has the roots

$$R_0 = 1, \quad R_1 = \frac{1}{i} = \frac{i}{i^2} = -i \quad \text{and} \quad R_2 = -\frac{1}{i} = i.$$

The three roots all differ, so the general solution is

$$z_t = b_0 + b_1(-i)^t + b_2 i^t \quad \text{for } t = 0, 1, \ldots.$$

Wrinting the complex numbers on the polar format we find (after some calculations) that the solution can alternatively be written as

$$z_t = b_0 + \widetilde{b}_1 \cos\left(\frac{\pi t}{2}\right) + \widetilde{b}_2 \sin\left(\frac{\pi t}{2}\right).$$

c) The equation is

$$z_t - 1.8z_{t-1} + 0.81z_{t-2} = 0 \quad \text{for} \quad t = 2, 3, \ldots.$$

To find the roots we need to solve

$$C(B) = 1 - 1.8B + 0.81B^2 = (1 - 0.9B)^2 = 0 \Rightarrow B = \frac{1}{0.9}.$$

Thereby $C(R^{-1}) = 0$ has one root $R_0 = 0.9$ with multiplicity $m_0 = 2$, and the general solution is

$$z_t = 0.9^t(b_0 + b_1 t) \quad \text{for } t = 0, 1, \ldots.$$

Problem 2. The zero-mean AR(1) model is given as

$$z_t = \varphi_1 z_{t-1} + a_t.$$

Multiplying with z_{t-k} on both sides and taking the expectation, we get

$$\mathrm{E}[z_{t-k} z_t] = \varphi_1 \mathrm{E}[z_{t-k} z_{t-1}] + \mathrm{E}[z_{t-k} a_t].$$

As z_{t-k} is uncorrelated with a_t for $k \geq 1$, we get for $k \geq 1$ that

$$\gamma_k = \varphi_1 \gamma_{k-1}.$$

Dividing with γ_0 on both sides of the equation we get a homogeneous difference equation of order 1 for the correlation function,

$$\rho_k = \varphi_1 \rho_{k-1} \quad \text{for } k = 1, 2, \ldots.$$

The general formula for ϕ_{kk} when $k \geq 2$ is

$$\phi_{kk} = \frac{\begin{vmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_1 \\ \rho_1 & \rho_0 & \rho_1 & \cdots & \rho_{k-3} & \rho_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & \rho_k \end{vmatrix}}{\begin{vmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & \rho_0 & \rho_1 & \cdots & \rho_{k-3} & \rho_{k-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & \rho_0 \end{vmatrix}}.$$

Using the above difference equation in the last column of the determinant in the numerator we get

$$\phi_{kk} = \frac{\begin{vmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \varphi_1\rho_0 \\ \rho_1 & \rho_0 & \rho_1 & \cdots & \rho_{k-3} & \varphi_1\rho_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & \varphi_1\rho_{k-1} \end{vmatrix}}{\begin{vmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & \rho_0 & \rho_1 & \cdots & \rho_{k-3} & \rho_{k-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & \rho_0 \end{vmatrix}}$$

and we see that the last column of the determinant in the numerator equals φ_1 times the first column of the same determinant. Thus, the determinant in the numerator equals zero and $\phi_{kk} = 0$ for $k \geq 2$.

Problem 3.

a)

- **i)** The model is given as

$$z_t = 0.6z_{t-1} + 0.3z_{t-2} + a_t.$$

For $k = 1, 2, \ldots$ this gives

$$\gamma_k = \mathrm{E}[z_{t-k}z_t] = \mathrm{E}[z_{t-k}(0.6z_{t-1} + 0.3z_{t-2} + a_t)]$$
$$= 0.6\mathrm{E}[z_{t-k}z_{t-1}] + 0.3\mathrm{E}[z_{t-k}z_{t-2}] + \mathrm{E}[z_{t-k}a_t]$$
$$= 0.6\gamma_{k-1} + 0.3\gamma_{k-2},$$

where we have used that z_{t-k} and a_t are independent when $k > 0$. Dividing by γ_0 we obtain an homogeneous difference equation for the correlation function

$$\rho_k - 0.6\rho_{k-1} - 0.3\rho_{k-2} = 0 \quad \text{for } k = 1, 2, 3, \ldots.$$

with initial conditions $\rho_0 = 1$ and (by using the above equation for $k = 1$) $\rho_1 - 0.6\rho_0 - 0.3\rho_{-1} = 0 \Rightarrow \rho_1 = \frac{6}{7}$. The difference equation for ρ_k can be formulated as

$$C(B)\rho_k = 0 \quad \text{where} \quad C(B) = 1 - 0.6B - 0.3B^2.$$

The equation $C(B) = 0$ has roots (detailed derivation not included) $B = -1 + \sqrt{\frac{13}{3}}$ and $B = -1 - \sqrt{\frac{13}{3}}$, so $C(R^{-1}) = 0$ will have the two roots

$$R_0 = \frac{1}{-1 + \sqrt{\frac{13}{3}}} = \frac{3}{10}\left(1 + \sqrt{\frac{13}{3}}\right) \quad \text{and} \quad R_1 = \frac{1}{-1 - \sqrt{\frac{13}{3}}} = \frac{3}{10}\left(1 - \sqrt{\frac{13}{3}}\right).$$

The general solution of the difference equation is thereby

$$\rho_k = b_0R_0^k + b_1R_1^k = \left(\frac{3}{10}\right)^k \left(b_0\left(1 + \sqrt{\frac{13}{3}}\right)^k + b_1\left(1 - \sqrt{\frac{13}{3}}\right)^k\right).$$

From the initial conditions we get

$$\rho_0 = b_0 + b_1 = 1$$

and

$$\rho_1 = \frac{3}{10}\left(b_0\left(1+\sqrt{\frac{13}{3}}\right)+b_1\left(1-\sqrt{\frac{13}{3}}\right)\right) = \frac{6}{7}.$$

Solving these two equations together with respect to b_0 and b_1 we get

$$b_0 = \frac{1}{2}+\frac{1}{14}\sqrt{39} \quad \text{and} \quad b_1 = \frac{1}{2}-\frac{1}{14}\sqrt{39}.$$

We thereby have

$$\rho_k = \left(\frac{3}{10}\right)^k \left(\left(\frac{1}{2}+\frac{1}{14}\sqrt{39}\right)\left(1+\sqrt{\frac{13}{3}}\right)^k + \left(\frac{1}{2}-\frac{1}{14}\sqrt{39}\right)\left(1-\sqrt{\frac{13}{3}}\right)^k\right),$$

which is valid for $k = 0, 1, \ldots$.

ii) No solution given here.

b) No solution given here.

c)

i) The model is given as

$$z_t = 0.6z_{t-1} + 0.3z_{t-2} + a_t.$$

This gives

$$\sigma^2 = \mathrm{E}[z_t z_t] = \mathrm{E}[z_t(0.6z_{t-1} + 0.3z_{t-2} + a_t)] = 0.6\gamma_1 + 0.3\gamma_2 + \mathrm{E}[z_t a_t],$$

where the last terms becomes

$$\mathrm{E}[z_t a_t] = \mathrm{E}[(0.6z_{t-1} + 0.3z_{t-2} + a_t)a_t] = \mathrm{E}[a_t^2] = \mathrm{Var}[a_t] = \sigma_a^2.$$

Using that $\gamma_k = \sigma^2 \cdot \rho_k$ we then get

$$\sigma^2 = 0.6\sigma^2\rho_1 + 0.3\sigma^2\rho_2 + \sigma_a^2,$$

and solving for σ^2 we get

$$\sigma^2 = \frac{\sigma_a^2}{1 - 0.6\rho_1 - 0.3\rho_2}.$$

From **a**) we have that $\rho_1 = \frac{6}{7} = 0.8571$ and $\rho_2 = 0.8143$, and thereby

$$\sigma^2 = \frac{\sigma_a^2}{1 - 0.6 \cdot 0.8571 - 0.3 \cdot 0.8143} = 4.1416\sigma_a^2.$$

Problem 4.

a) We should consider the AR(2) model

$$z_t = z_{t-1} + \alpha z_{t-2} + a_t.$$

Then

$$\varphi(B) = 1 - B - \alpha B^2.$$

To check for stationarity we need to find the roots of $\varphi(B) = 0$, which are

$$B_1 = \frac{-1 + \sqrt{1+4\alpha}}{2\alpha} \quad \text{and} \quad B_2 = \frac{-1 - \sqrt{1+4\alpha}}{2\alpha}.$$

If $1+4\alpha \geq 0 \Leftrightarrow \alpha \geq -\frac{1}{4}$ the roots are real and we see that we then always have $|B_1| \leq |B_2|$, so to get a stationary model it is sufficient that

$$|B_1| = \left| \frac{\sqrt{4\alpha+1} - 1}{2\alpha} \right| > 1.$$

When $\alpha \in [-1/4, 0)$ both the numerator and denominator in the expression for B_1 are negative so the equirement becomes

$$-(\sqrt{4\alpha+1} - 1) > -2\alpha$$

$$\sqrt{4\alpha+1} < 2\alpha + 1$$

$$4\alpha + 1 < (2\alpha - 1)^2 = 4\alpha^2 - 4\alpha + 1 \Rightarrow 4\alpha^2 > 0,$$

which is always fulfilled for $\alpha \in [-1/4, 0)$. When $\alpha > 0$ both the numerator and denominator in the expression for B_1 are positive and the requirement becomes

$$\sqrt{4\alpha+1} - 1 > 2\alpha$$

$$\sqrt{4\alpha+1} > 2\alpha + 1$$

$$4\alpha + 1 > (2\alpha + 1)^2 = 4\alpha^2 + 4\alpha + 1 \Rightarrow 0 > 4\alpha^2,$$

which is never fulfilled for $\alpha > 0$. One can also note that $\alpha = 0$ gives a (non-stationary) AR(1) process, so $\alpha = 0$ is not allowed.

It remains to check the possibility $1 + 4\alpha < 0 \Leftrightarrow \alpha < -\frac{1}{4}$ which gives complex roots,

$$|B_1|^2 = |B_2|^2 = \left| -\frac{1}{2\alpha} \pm i \frac{\sqrt{-(1+4\alpha)}}{2\alpha} \right|^2$$

$$= \left(-\frac{1}{2\alpha} \right)^2 + \frac{-(1+4\alpha)}{(2\alpha)^2} = \frac{1 - 1 - 4\alpha}{4\alpha^2} = -\frac{1}{\alpha}.$$

The requirement for stationarity is

$$|B_1|^2 > 1 \Leftarrow -\frac{1}{\alpha} > 1 \Leftarrow 1 > -\alpha \Leftarrow -1 < \alpha \Leftarrow \alpha > -1.$$

Thus the model is stationary with complex roots if $\alpha \in (-1, -1/4)$, and it is stationary with real roots if $\alpha \in [-1/4, 0)$. So the model is stationary if $\alpha \in (-1, 0)$.

b) No solution given here.