TMA4285 Time series models
Solution to exercise 4, autumn 2014

Problem 3.8.
a) The model is

$$z_t = a_t + 1.2a_{t-1} - 0.5a_{t-2}$$

For $k = 0, 1, \dots$ we get

$$\gamma_k = \mathrm{E}[z_t z_{t+k}] = \mathrm{E}[(a_t + 1.2a_{t-1} - 0.5a_{t-2})(a_{t+k} + 1.2a_{t+k-1} - 0.5a_{t+k-2})]$$
$$= \begin{cases} \sigma_a^2(1 + 1.2^2 + 0.5^2) = 2.69\sigma_a^2 & \text{for } k = 0, \\ \sigma_a^2(-1.2 - 1.2 \cdot 0.5) = -1.8\sigma_a^2 & \text{for } k = 1, \\ \sigma_a^2 \cdot 0.5 = 0.5\sigma_a^2 & \text{for } k = 2, \\ 0 & \text{for } k = 3, 4, \dots. \end{cases}$$

Thereby the ACF becomes

$$\rho_k = \begin{cases} 1 & \text{for } k = 0, \\ \frac{-1.8}{2.69} = -0.67 & \text{for } k = 1, \\ \frac{0.5}{2.69} = 0.19 & \text{for } k = 2, \\ 0 & \text{for } k = 3, 4, \dots. \end{cases}$$

c) Using the results in **a**) we get

$$\phi_{11} = \rho_1 = -0.67,$$

$$\phi_{22} = \frac{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & \rho_2 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & -0.67 \\ -0.67 & 0.19 \end{vmatrix}}{\begin{vmatrix} 1 & -0.67 \\ -0.67 & 1 \end{vmatrix}} = \frac{-0.2589}{0.5511} = -0.47,$$

$$\phi_{33} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & \rho_3 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & -0.67 & -0.67 \\ -0.67 & 1 & 0.19 \\ 0.19 & -0.67 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -0.67 & 0.19 \\ -0.67 & 1 & -0.67 \\ 0.19 & -0.67 & 1 \end{vmatrix}} = \frac{-0.0704}{0.2367} = -0.30.$$

Problem 3.9. For the MA(1) model

$$z_t = a_t - \theta_1 a_{t-1}$$

we know that the ACF is given by

$$\rho_k = \begin{cases} 1 & \text{for } k = 0, \\ -\frac{\theta_1}{1+\theta_1^2} & \text{for } k = 1, \\ 0 & \text{for } k = 2, 3, \ldots. \end{cases}$$

By choosing

$$\rho_1 = -\frac{\theta_1}{1+\theta_1^2} = 0.25 \Rightarrow \theta = -2 \pm \sqrt{3}$$

we get a model with the required ACF. For the model to be invertible we also need $|\theta_1| < 1$, so we must use $\theta_1 = -2 + \sqrt{3} = -0.268$.

Problem 3.10.
a) For the MA(2) model

$$z_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$

we know that the autocovariance function is given by

$$\gamma_k = \begin{cases} (1+\theta_1^2+\theta_2^2)\sigma_a^2 & \text{for } k = 0, \\ -\theta_1(1-\theta_2)\sigma_a^2 & \text{for } k = 1, \\ -\theta_2\sigma_a^2 & \text{for } k = 2, \\ 0 & \text{for } k = 3, 4, \ldots. \end{cases}$$

By choosing

$$(1+\theta_1^2+\theta_2^2)\sigma_a^2 = 10, \quad -\theta_1(1-\theta_2)\sigma_a^2 = 0 \text{ and } -\theta_2\sigma_a^2 = -4$$

we get the autocovariance function we want. From the second equation we see that we must have $\theta_1 = 0$ or $\theta_2 = 1$. We first try $\theta_1 = 0$. Then the third equation give $\theta_2 = \frac{4}{\sigma_a^2}$, and inserting this into the first equation we get

$$\left(1 + \left(\frac{4}{\sigma_a^2}\right)^2\right)\sigma_a^2 = 10 \Rightarrow \sigma_a^2 = 2 \text{ or } \sigma_a^2 = 8.$$

Choosing $\theta_2 = 1$ the third equation gives $\sigma_a^2 = 4$, and inserting this into the first equation we get

$$\left(1 + \theta_1^2 + 1^2\right) \cdot 4 = 10 \Rightarrow \theta_1 = \pm\frac{\sqrt{2}}{2}.$$

Thereby we have found as much as four models with the required autocovariance function, namely

$$z_t = a_t - 2a_{t-2} \text{ with } \sigma_a^2 = 2,$$

$$z_t = a_t - \frac{1}{2}a_{t-2} \text{ with } \sigma_a^2 = 8,$$

$$z_t = a_t - \frac{\sqrt{2}}{2} a_{t-1} - a_{t-2} \quad \text{with } \sigma_a^2 = 4,$$

and

$$z_t = a_t + \frac{\sqrt{2}}{2} a_{t-1} - a_{t-2} \quad \text{with } \sigma_a^2 = 4.$$

b) An MA(2) process is always stationary, so all four processes found in **a**) are stationary. To check whether the models are invertible we need to find the roots of $\theta(B) = 0$. For the first model we get

$$1 - 2B^2 = 0 \Rightarrow B = \pm\frac{\sqrt{2}}{2}.$$

The two roots are not outside the unit circle, so the model is not invertible. For the second model we get

$$1 - \frac{1}{2}B^2 = 0 \Rightarrow B = \pm\sqrt{2}.$$

Both roots are outside the unit circle, so the model is invertible. For the third model we get

$$1 - \frac{\sqrt{2}}{2}B - B^2 = 0 \Rightarrow B = -\sqrt{2} \quad \text{or} \quad B = \frac{\sqrt{2}}{2}.$$

The second root is not outside the unit circle, so the model is not invertible. For the fourth model we get

$$1 + \frac{\sqrt{2}}{2}B - B^2 = 0 \Rightarrow B = -\frac{\sqrt{2}}{2} \quad \text{or} \quad B = \sqrt{2}.$$

Again one of the roots are not outside the unit circle, so this model is not invertible either. Thus, only one of the models found in **a**) is invertible, namely

$$z_t = a_t - \frac{1}{2} a_{t-2} \quad \text{with } \sigma_a^2 = 8.$$

Problem 3.11.
a) All MA(2) processes are stationary, so the current one is thereby also stationary.

b) We need to find the roots of $\theta(B) = 0$,

$$1 - 0.1B + 0.21B^2 = 0 \Rightarrow B = 0.2381 \pm 2.1692i.$$

We see that both roots are outside the unit circle, so the model is invertible.

c) No solution given here.

Problem 3.12. One can simulate in R with the command

$$x = \text{arima.sim(model = list(order = c}(0, 0, 2), \text{ma = c}(-1.2, 0.5)), \text{n} = 100),$$

but you should note that the the MA model in R is defined with opposite sign for the MA coefficients θ_1 and θ_2 relative to what we do in the lectures and in Wei (2006). One can estimate the autocorrelation and partial autocorrelation functions with the R functions acf and pacf, respectively. To estimate ρ_k and ϕ_{kk} for the required values of k one can use the option "lag.max=20" in the acf and pacf functions, for instance

$$\text{acf}(\text{x}, \text{lag.max} = 20).$$

Problem 3.13. No solution given here.

Problem 8. First note that reversing the order of the coefficients in a MA(q)-model don't lead to any change in the autocorrelation function. By choosing a suitable value of $\sigma_a^{2'}$ in the reparameterized model, the autocovariance function will also be equal to that of the original non-invertible model. Since the distribution of the data is the same, these alternative parameter values only represents alternative parameterizations of the same model.

Let $\theta(B) = (1 - R_1 B)(1 - R_2 B) = 1 - (R_1 + R_2)B + R_1 R_2 B^2$ represent the moving average polynomial of the non-invertible model. If both roots are inside the unit circle, we have $|R_1| > 1$ and $|R_2| > 1$. The reparameterized model obtained by reversing the order of the MA-coefficients is given by

$$\theta'(B) = (R_1 R_2 - (R_1 + R_2)B + B^2)/(R_1 R_2) \tag{1}$$

if also rescaling all coefficients such that the convention $\theta_0 = 1$ is satisfied. This can be rewritten as

$$\theta'(B) = (R_1 - B)(R_2 - B)/(R_1 R_2) \tag{2}$$

from which it is clear that the roots of the reparameterized model $B_1' = R_1$ and $B_2' = R_2$ are both outside the unitcircle. The reparameterized model is thus invertible.

Update It turns out that any non-invertible MA(q) model can be made invertible through reparameterization except if the roots are exactly on the unit circle. Consider the model MA(q)-model

$$\theta(B) = \prod_{i=1}^{q} (1 - R_i B). \tag{3}$$

This has autocovariancegenerating function (see 2.6.8-2.6.9 in Wei)

$$\gamma(B) = \sigma_a^2 \prod_{i=1}^{q} (1 - R_i B)(1 - R_i \frac{1}{B}). \tag{4}$$

Suppose that the first root is inside the unit circle, that is, $|B_1| < 1$ and $|R_1| > 1$ such that model is non-invertible.

Consider the model obtained by changing R_1 to $R_1' = 1/R_1$ and the white noise variance to $\sigma_a^{2'} = \sigma_a^2 R_1^2$. This model with different parameter values is invertible. Is it still the same model? The autocovariance generating function becomes

$$\gamma(B) = \sigma_a'^2 (1 - R_i' B)(1 - R_i' \frac{1}{B}) \prod_{i=2}^{q} (1 - R_i B)(1 - R_i \frac{1}{B}) \tag{5}$$

$$= \sigma_a^2 R_1^2 (1 - \frac{1}{R_1} B)(1 - \frac{1}{R_1} \frac{1}{B}) \prod_{i=2}^{q} (1 - R_i B)(1 - R_i \frac{1}{B}) \tag{6}$$

$$= \sigma_a^2 R_1^2 (1 + \frac{1}{R_1^2} - \frac{1}{R_1} B - \frac{1}{R_1} B) \prod_{i=2}^{q} (1 - R_i B)(1 - R_i \frac{1}{B}) \tag{7}$$

$$= \sigma_a^2 (R_1^2 + 1 - R_1 B - R_1 B) \prod_{i=2}^{q} (1 - R_i B)(1 - R_i \frac{1}{B}) \tag{8}$$

which equals the autocovariance generating function (4) of (3). Thus the two models are equivalent representations of the same stochastic process and just differ in terms of their parameterizations.