NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for
Solution: TMA4285 Time series models

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Examination date: December 7th 2013

Examination time (from–to): 09:00–13:00

Permitted examination support material: C:

- Calculator CITIZEN SR-270X, CITIZEN SR-270X College or HP30S.
- Statistiske tabeller og formler, Tapir forlag.
- K. Rottman: Matematisk formelsamling.
- One yellow, stamped A5 sheet with own handwritten formulas and notes.

Other information:
Note that all answers should be justified.
In your solution you can use English and/or Norwegian.

Language: English

Number of pages: 3

Number pages enclosed: 0

Checked by:

Date Signature

Problem 1

a) Both ts1 and ts2 seem to be stationary , so one should have $d = 0$ for both time series.

The acf for ts1 seems to be a damped sine wave, whereas the pacf cuts off after lag 3. This behaviour is consistent with an AR(3) model, so one should start with $\underline{\underline{p = 3,\ d = 0 \text{ and } q = 0}}$.

The acf for ts2 cuts off after lag 1, whereas the pacf seems to decay exponentially. This behaviour is consistent with an MA(1) model, so one should start with $\underline{\underline{p = 0,\ d = 0 \text{ and } q = 1}}$.

Problem 2

a) For a time series z_t to be second-order stationary one must have

$$F_{z_{t_1}, z_{t_2}}(x_1, x_2) = F_{z_{t_1+k}, z_{z_2+k}}(x_1, x_2),$$

for all t_1, t_2, k and all x_1, x_2.

For a time series z_t to be covariance stationary all first and second order moments must exist and be time invariant.

A time series process z_t which is covariance stationary, does not need to be second-order stationary. Even if the two first moments is time invariant, the joint distribution does not need to be time invariant.

A time series process z_t which is second-order stationary does not need to be covariance stationary. If all the first and second order moments exist for a second-order stationary process, then it is also covariance stationary, but the first and second order moments do not need to exist for a second-order stationary process.

b) The process is invertible if and only if all the roots of $\theta(B) = 1 - \theta_1 B = 0$ are outside the unit circle. The current model has only one root, which is $B = 1/\theta_1$ so thereby the process is invertible if

$$\left| \frac{1}{\theta_1} \right| > 1 \Leftrightarrow \underline{\underline{\theta \in (-1, 1)}}.$$

The model is covariance stationary if and only if the roots of $\varphi(B) = 1 - 1.7B + 0.72B^2 = 0$ are outside the unit circle. The roots in the current model become

$$B = \frac{1.7 \pm \sqrt{1.7^2 - 4 \cdot 0.72}}{2 \cdot 0.72} = \frac{1.7 \pm \sqrt{0.01}}{1.44} \Rightarrow B = 1.25 \text{ or } B = 1.11.$$

Thus, the model is covariance stationary.

c) From the model we have that

$$z_{t+k} = \varphi_1 z_{t+k-1} + \varphi_2 z_{t+k-2} + a_{t+k} - \theta_1 a_{t+k-1}.$$

Thereby we get for $k = 0, 1, 2, \ldots$,

$$\begin{aligned}
\gamma_k &= \mathrm{E}[z_{t+k} z_t] = \mathrm{E}[(\varphi_1 z_{t+k-1} + \varphi_2 z_{t+k-2} + a_{t+k} - \theta_1 a_{t+k-1}) z_t] \\
&= \varphi_1 \mathrm{E}[z_{t+k-1} z_t] + \varphi_2 \mathrm{E}[z_{t+k-2} z_t] + \mathrm{E}[a_{t+k} z_t] - \theta_1 \mathrm{E}[a_{t+k-1} z_t] \\
&= \varphi_1 \gamma_{k-1} + \varphi_2 \gamma_{k-2} + \mathrm{E}[a_{t+k} z_t] - \theta_1 \mathrm{E}[a_{t+k-1} z_t].
\end{aligned}$$

For $k = 2, 3, \ldots$ we have that both $t + k$ and $t + k - 1$ is larger that t and thereby a_{t+k} and a_{t+k-1} are both uncorrelated with z_t, so $\mathrm{E}[a_{t+k} z_t] = \mathrm{E}[a_{t+k}]\mathrm{E}[z_t] = 0 \cdot \mathrm{E}[z_t] = 0$ and $\mathrm{E}[a_{t+k-1} z_t] = \mathrm{E}[a_{t+k-1}]\mathrm{E}[z_t] = 0 \cdot \mathrm{E}[z_t] = 0$. Thus we have found the homogeneous difference equation

$$\gamma_k - \varphi_1 \gamma_{k-1} - \varphi_2 \gamma_{k-2} = 0 \quad \text{for } k = 2, 3, \ldots. \tag{1}$$

To find the initial conditions we need to consider the above expression for γ_k for $k < 2$. We start with $k = 0$,

$$\begin{aligned}
\gamma_0 &= \varphi_1 \gamma_{-1} + \varphi_2 \gamma_{-2} + \mathrm{E}[a_t z_t] - \theta_1 \mathrm{E}[a_{t-1} z_t] \\
&= \varphi_1 \gamma_{-1} + \varphi_2 \gamma_{-2} + \mathrm{E}[a_t(\varphi_1 z_{t-1} + \varphi_2 z_{t-2} + a_t - \theta_1 a_{t-1})] \\
&\quad -\theta_1 \mathrm{E}[a_{t-1}(\varphi_1 z_{t-1} + \varphi_2 z_{t-2} + a_t - \theta_1 a_{t-1})] \\
&= \varphi_1 \gamma_{-1} + \varphi_2 \gamma_{-2} + \mathrm{E}[a_t \cdot a_t] - \theta_1(\varphi_1 \mathrm{E}[a_{t-1} z_{t-1}] - \theta_1 \mathrm{E}[a_{t-1} \cdot a_{t-1}]) \\
&= \varphi_1 \gamma_{-1} + \varphi_2 \gamma_{-2} + \sigma_a^2 - \theta_1(\varphi_1 \sigma_a^2 - \theta_1 \sigma_a^2).
\end{aligned}$$

Thus, using that γ_k is a symmetric function, the first equation becomes

$$\gamma_0 = \varphi_1 \gamma_1 + \varphi_2 \gamma_2 + \sigma_a^2 (1 - \theta_1 \varphi_1 + \theta_1^2). \tag{2}$$

Doing the same for $k = 1$ we get

$$\gamma_1 = \varphi_1 \gamma_0 + \varphi_2 \gamma_{-1} + \mathrm{E}[a_{t+1} z_t] - \theta_1 \mathrm{E}[a_t z_t].$$

Again using what we found above, namely that $\mathrm{E}[a_{t+1} z_t] = 0$ and $\mathrm{E}[a_t z_t] = \sigma_a^2$ the second equation becomes

$$\gamma_1 = \varphi_1 \gamma_0 + \varphi_2 \gamma_1 - \theta_1 \sigma_a^2. \tag{3}$$

As (2) and (3) includes both γ_0, γ_1 and γ_2 we need a third equation, which we get for $k = 2$. This, however, we get directly from (1),

$$\gamma_2 = \varphi_1 \gamma_1 + \varphi_2 \gamma_0. \tag{4}$$

Thereby we have three equations, (2), (3) and (4), with three unknowns, γ_0, γ_1 and γ_2, which can be solved to find the initial conditions.

d) We have

$$\varphi(B)z_t = \theta(B)a_t \Rightarrow \frac{\varphi(B)}{\theta(B)}z_t = a_t.$$

Thereby we must have

$$\psi(B) = \frac{\varphi(B)}{\theta(B)} \Rightarrow \theta(B)\psi(B) = \varphi(B),$$

where $\psi(B) = 1 - \psi_1 B - \psi_2 B^2 - \dots$. Thereby,

$$(1-\theta_1 B)(1-\psi_1 B - \psi_2 B^2 - \psi_3 B^3 - \dots) = 1 - \varphi_1 B - \varphi_2 B^2.$$

By expanding on the left hand side of this equation and setting equal coefficients in front of the same power of B, we sequentially get

$$\begin{aligned}
B^1: &\quad -\psi_1 - \theta_1 = -\varphi_1 \Rightarrow= \psi_1 = \varphi_1 - \theta_1, \\
B^2: &\quad -\psi_2 + \theta_1\psi_1 = -\varphi_2 \Rightarrow \psi_2 = \varphi_2 + \theta_1\psi_1 = \varphi_2 + \theta_1(\varphi_1 - \theta_1), \\
B^3: &\quad -\psi_3 + \theta_1\psi_2 = 0 \Rightarrow \psi_3 = \theta_1\psi_2 = \theta_1(\varphi_2 + \theta_1(\varphi_1 - \theta_1)), \\
&\quad \vdots \\
B^k: &\quad -\psi_k + \theta_1\psi_{k-1} = 0 \Rightarrow \psi_k = \theta_1\psi_{k-1} = \theta_1^{k-2}(\varphi_2 + \theta_1(\varphi_1 - \theta_1)).
\end{aligned}$$