TMA4285 Time series models
Solution to exercise 6, autumn 2015

Problem 4.1. We have the model

$$(1-B)^2 z_t = (1 - 0.3B - 0.5B^2)a_t.$$

a) The model is not stationary because the polynomial $(1-B)^2$ has (two) roots at $B=1$, which is not outside the unit circle.
b) If $w_t = (1-B)^2 z_t$ we have

$$w_t = (1 - 0.3B - 0.5B^2)a_t,$$

which is an MA(2) model. The model for w_t is stationary because MA(q) models are always stationary.
c) We have

$$w_t = a_t - 0.3a_{t-1} - 0.5w_{t-2}.$$

As the a_t's are uncorrelated this gives

$$\gamma_k = \mathrm{E}[w_t w_{t+k}] = \mathrm{E}[(a_t - 0.3a_{t-1} - 0.5a_{t-2})(a_{t+k} - 0.3a_{t+k-1} - 0.5a_{t+k-2})]$$
$$= \begin{cases} \sigma_a^2(1+0.3^2+0.5^2) & \text{if } k=0, \\ \sigma_a^2(-0.3+0.3\cdot 0.5) & \text{if } k=1, \\ \sigma_a^2(-0.5) & \text{if } k=2, \\ 0 & \text{if } k\geq 3. \end{cases}$$

Thereby we get the autocorrelation function to be

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} 1 & \text{if } k=0, \\ \frac{-0.3+0.3\cdot 0.5}{1+0.3^2+0.5^2} = -0.11 & \text{if } k=1, \\ \frac{-0.5}{1+0.3^2+0.5^2} = -0.37 & \text{if } k=2, \\ 0 & \text{if } k\geq 3. \end{cases}$$

Problem 4.2a We have the model

$$(1-B)^2 z_t = (1 - 0.81B + 0.38B^2)a_t.$$

We can write this on an AR(∞) form,

$$\pi(B)z_t = a_t,$$

by setting

$$\pi(B) = \frac{(1-B)^2}{1-0.81B+0.38B^2} \Leftrightarrow \pi(B)(1-0.81B+0.38B^2) = (1-B)^2.$$

This gives

$$(1-\pi_1 B - \pi_2 B^2 - \pi_3 B^2 - \ldots)(1-0.81B+0.38B^2) = 1-2B+B^2.$$

Multiplying out and identifying coefficients in from of the same powers of B we get,

$$\begin{array}{rc} B^1: & -0.81-\pi_1 = -2 \Rightarrow \pi_1 = 1.19 \\ B^2: & -\pi_2 + 0.81\pi_1 + 0.38 = 1 \Rightarrow \pi_2 = 0.81\pi_1 + 0.38 - 1 = 0.3439 \\ B^3: & -\pi_3 + 0.81\pi_2 - 0.38\pi_1 = 0 \Rightarrow \pi_3 = 0.81\pi_2 - 0.38\pi_1 = -0.173641 \\ \vdots & \\ B^k: & -\pi_k + 0.81\pi_{k-1} - 0.38\pi_{k-2} = 0 \text{ for } k = 3,4,\ldots \end{array}$$

To find π_k for $k = 3, 4, \ldots$ we thereby need to solve the homogeneous difference equation

$$(1-0.81B+0.38B^2)\pi_k = 0 \quad \text{for } k = 3,4,\ldots,$$

with initial conditions $\pi_1 = 1.19$ and $\pi_2 = 0.3439$. The roots of

$$1-0.81B+0.38B^2 = 0$$

are $B = 1.0658 + 1.2230i$ and $B = 1.0658 - 1.2230i$. For $R = B^{-1}$ this gives $R = 1/(1.0658 + 1.2230i) = 0.4050 - 0.4647i$ and $R = 1/(1.0658 - 1.2230i) = 0.4050 + 0.4647i$. The general solution of the homogeneous difference equation is thereby

$$\pi_k = b_1(0.4050 - 0.4647i)^k + b_2(0.4050 + 0.4647i)^k.$$

Using the initial conditions we get equations for b_1 and b_2,

$$b_1(0.4050 - 0.4647i)^1 + b_2(0.4050 + 0.4647i)^1 = 1.19,$$

$$b_1(0.4050 - 0.4647i)^2 + b_2(0.4050 + 0.4647i)^2 = 0.3439.$$

Solving these equations with respect to b_1 and b_2 we get $b_1 = 0.8159 + 0.5694i$ and $b_2 = 0.8159 - 0.5694i$. Thereby the π_k coefficients becomes

$$\pi_k = (0.8159 + 0.5694i) \cdot (0.4050 - 0.4647i)^k + (0.8159 - 0.5694i) \cdot (0.4050 + 0.4647i)^k$$

for $k = 1, 2, \ldots$. One should note that all π_k are in fact real even though the expression above suggests it is complex. As discussed in pages 28-29 i Wei (2006) one may rewrite the expression for π_k using real numbers only.

Problem 4.5. When $z_a = a_t$ and $w_t = z_t - z_{t-1}$ we get

$$w_t = a_t - a_{t-1}.$$

The autocovariance function for w_t becomes

$$\gamma_k = \mathrm{E}[w_t w_{t+k}] = \mathrm{E}[(a_t - a_{t-1})(a_{t+k} - a_{t+k-1})]$$

$$= \begin{cases} \mathrm{E}[(a_t - a_{t-1})^2] = \mathrm{E}[a_t^2 - 2a_t a_{t-1} + a_{t-1}^2] = 2\sigma_a^2 & \text{for } k = 0, \\ \mathrm{E}[(a_t - a_{t-1})(a_{t+1} - a_t)] = -\mathrm{E}[a_t^2] = -\sigma_a^2 & \text{for } k = 1, \\ 0 & \text{for } k \geq 2. \end{cases}$$

The autocorrelation function thereby becomes

$$\rho_k = \begin{cases} 1 & \text{for } k = 0, \\ -\frac{1}{2} & \text{for } k = 1, \\ 0 & \text{for } k \geq 2. \end{cases}$$

The partial autocorrelation function becomes

$$\phi_{11} = \rho_1 = -\frac{1}{2},$$

$$\phi_{22} = \frac{\begin{vmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{vmatrix}} = \frac{-\frac{1}{4}}{1 - \frac{1}{4}} = -\frac{1}{3},$$

and for $k \geq 3$

$$\phi_{kk} = \frac{A_k}{B_k},$$

where

$$A_k = \begin{vmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -\frac{1}{2} & 0 \end{vmatrix}$$

and

$$B_k = \begin{vmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & \cdots & -\frac{1}{2} & 1 \end{vmatrix}.$$

Expanding A_k recursively along the last line we obtain

$$A_k = -\left(-\frac{1}{2}\right) A_{k-1} = \frac{1}{2} A_{k-1} = \frac{1}{2^2} A_{k-2} = \ldots = \frac{1}{2^{k-2}} A_2 = \frac{1}{2^{k-2}} \cdot \left(-\frac{1}{4}\right) = -\frac{1}{2^k}.$$

The matrix defining B_k is a tridiagonal matrix and then the determinants can be computed recursively, see for example http://en.wikipedia.org/wiki/Tridiagonal_matrix. Applying this on B_k we get

$$B_k = 1 \cdot B_{k-1} - \left(-\frac{1}{2}\right)^2 B_{k-2} = B_{k-1} - \frac{1}{4} B_{k-2}.$$

This is in fact a homogeneous difference equation of order 2 for B_k, and we can easily find initial conditions $B_2 = \frac{3}{4}$ and $B_3 = \frac{1}{2}$. Solving the difference equation we find the general solution for B_k,

$$B_k = \left(\frac{1}{2}\right)^k (b_0 + b_1 k).$$

Using the initial conditions we find that $b_0 = 1$ and $b_1 = 1$, so

$$B_k = \frac{1+k}{2^k},$$

and thereby

$$\phi_{kk} = \frac{-\frac{1}{2^k}}{\frac{1+k}{2^k}} = -\frac{1}{1+k},$$

which is valid for $k = 1, 2, \ldots$.

The model for w_t is

$$w_t = (1 - B) a_t,$$

and we see that the root of $1 - B = 0$ does not lie outside the unit circle. Thereby the model is not invertable, so an AR representation of the model is not possible.

So by overdifferencing we end up with a non-invertable MA(1) process.

Problem 5.2. The forecast error can be expressed as

$$e_n(l) = \sum_{k=0}^{l-1} \psi_k a_{n+l-k}$$

where the coefficients ψ_k are given from

$$\sum_{i=0}^{m} \pi_{m-i} \psi_i = 0 \quad \text{for } m = 1, \ldots, l-1$$

and the π_i's are the coefficients in the AR(∞) representation on the model. In particular one should note that the ψ_k coefficients are not a function of l.

a)

$$\text{Cov}[e_n(l), e_{n-j}(l)] = \text{Cov}\left[\sum_{k=0}^{l-1} \psi_k a_{n+l-k}, \sum_{r=0}^{l-1} \psi_r a_{n-j+l-r}\right]$$

$$= \sum_{k=0}^{l-1}\sum_{r=0}^{l-1} \psi_k \psi_r \text{Cov}[a_{n+l-k}, a_{n-j+l-r}] = \sigma_a^2 \sum_{k=j}^{l-1} \psi_k \psi_{k-j}$$

since $\text{Cov}[a_{n+l-k}, a_{n-j+l-r}] = \sigma_a^2$ if $n+l-k = n-j+l-r \Leftrightarrow r = k-j$, and zero otherwise.

b)

$$\text{Cov}[e_n(l), e_n(l+j)] = \text{Cov}\left[\sum_{k=0}^{l-1} \psi_k a_{n+l-k}, \sum_{r=0}^{l+j-1} \psi_r a_{n+l+j-r}\right]$$

$$= \sum_{k=0}^{l-1}\sum_{r=0}^{l+j-1} \psi_k \psi_r \text{Cov}[a_{n+l-k}, a_{n+l+j-r}] = \sigma_a^2 \sum_{k=0}^{l-1} \psi_k \psi_{k+j}$$

since $\text{Cov}[a_{n+l-k}, a_{n-j+l-r}] = \sigma_a^2$ if $n+l-k = n+l+j-r \Leftrightarrow r = k+j$, and zero otherwise.

Problem 5.5 The given model can be expressed as

$$z_t = 2.4z_{t-1} - 1.88z_{t-2} + 0.48z_{t-3} + a_t.$$

Substituting t with $n+l$ and taking the conditional mean given $z_t, t \leq n$ we get for $l > 0$

$$\widehat{z}_n(l) = 2.4\widehat{z}_n(l-1) - 1.88\widehat{z}_n(l-2) + 0.48\widehat{z}_n(l-3).$$

a) From this and the given oberved values we get

$$\widehat{z}_n(1) = 2.4z_n - 1.88z_{n-1} + 0.48z_{n-2} = 779.6,$$

$$\widehat{z}_n(2) = 2.4\widehat{z}_n(1) - 1.88z_n + 0.48z_{n-1} = 736.64,$$

$$\widehat{z}_n(3) = 2.4\widehat{z}_n(2) - 1.88\widehat{z}_n(1) + 0.48z_n = 686.288.$$

b) The forcast variances are given as

$$\text{Var}[z_{n+l} - \widehat{z}_n(l)] = \sigma_a^2 \sum_{k=0}^{l-1} \psi_k^2,$$

where the coefficients ψ_k are given from

$$\sum_{i=0}^{m} \pi_{m-i}\psi_i = 0 \Rightarrow \psi_m = \sum_{i=0}^{m-1} \pi_{m-i}\psi_i \quad \text{for } m = 1, \ldots, l-1$$

with $\psi_0 = 1$, and the π_i's are the coefficients in the AR(∞) representation of the model. Our model is already on the AR form and we have $\pi_0 = -1$, $\pi_1 = 2.4$, $\pi_2 = -1.88$, $\pi_3 = 0.48$, $\pi_k = 0$ for $k = 4, 5, \ldots$. This gives

$$\psi_0 = 1, \psi_1 = \pi_1 \psi_0 = 2.4,$$

$$\psi_2 = \pi_2 \psi_0 + \pi_1 \psi_1 = -1.88 + 2.4 \cdot 2.4 = 3.88$$

and

$$\psi_3 = \pi_3 \psi_0 + \pi_2 \psi_1 + \pi_1 \psi_2 = 0.48 - 1.88 \cdot 2.4 + 2.4 \cdot 3.88 = 5.28,$$

and thereby

$$\mathrm{Var}[z_{n+1} - \widehat{z}_n(1)] = 58\ 000 \cdot 1^2,$$

$$\mathrm{Var}[z_{n+2} - \widehat{z}_n(2)] = 58\ 000 \cdot (1^2 + 2.4^2) = 392\ 080,$$

and

$$\mathrm{Var}[z_{n+3} - \widehat{z}_n(3)] = 58\ 000 \cdot (1^2 + 2.4^2 + 3.88^2) = 1\ 265\ 235.2.$$

The three 95% prediction intervals then becomes $[779.6 - 1.96 \cdot \sqrt{58\ 000}, 779.6 + 1.96 \cdot \sqrt{58\ 000}] = [307.6, 1\,251.6]$, $[736.64 - 1.96\sqrt{392\ 080}, 736.64 + 1.96\sqrt{392\ 080}] = [-490.6, 1\,963.9]$ and $[686.288 - 1.96\sqrt{1\ 265\ 235.2}, 686.288 + 1.96\sqrt{1\ 265\ 235.2}] = [-1\ 518.4, 2\ 890.9]$.

c) No solution given here.