TMA4285 Time series models
Solution to exercise 10, autumn 2018

November 28, 2018

Problem 7.1

From the process we have $Z_t = \sqrt{h_t} e_t$ so that $Z_t^4 = h_t^2 e_t^4$.

$$E(Z_t^4) = E(e_t^4 h_t^2) = E(e_t^4)E(h_t^2) = (Var(e_t^2) + E(e_t^2)^2)E(h_t^2) = 3E(h_t^2)$$

since $Var(e_t^2) = 2$ and $E(e_t^2) = 1$. We use that $h_t = Z_t^2/e_t^2 = (\alpha_0 e_t^2 + \alpha_1 e_t^2 Z_{t-1}^2)/e_t^2 = \alpha_0 + \alpha_1 Z_{t-1}^2$. Then

$$\begin{aligned}
E(Z_t^4) &= 3E(\alpha_0^2 + 2\alpha_0\alpha_1 Z_{t-1}^2 + \alpha_1^2 Z_{t-1}^4) \\
E(Z_t^4) &= 3\alpha_0^2 + 6\alpha_0\alpha_1 \frac{\alpha_0}{1-\alpha_1} + 3\alpha_1^2 E(Z_{t-1}^4) \\
E(Z_t^4)(1 - 3\alpha_1^2) &= \frac{(1-\alpha_1)3\alpha_0^2 + 6\alpha_0^2\alpha_1}{1-\alpha_1} \\
E(Z_t^4) &= \frac{3\alpha_0^2(1+\alpha_1)}{(1-3\alpha_1^2)(1-\alpha_1)}
\end{aligned}$$

where we have used that $E(Z_t^2) = \frac{\alpha_0}{1-\alpha_1}$. This equation can only be satisfied if $0 < \alpha_1^2 < 1/3$, which becomes the condition for finite expectation.

Problem 7.2

We insert $Z_t^2 = h_t e_t^2$ into Y_t

$$Y_t = \frac{h_t e_t^2}{\alpha_0} = e_t^2(1 + \frac{1}{\alpha_0}\sum_{i=1}^{p} \alpha_i Z_{t-i}^2) = e_t^2(1 + \frac{1}{\alpha_0}\sum_{i=1}^{p} \alpha_0\alpha_i Y_{t-i}) = e_t^2(1 + \sum_{i=1}^{p} \alpha_i Y_{t-i})$$

where we have used that $Y_{t-i}\alpha_0 = Z_{t-i}^2$.

Next, we are supposed to deduce that $\{Y_t\}$ has the same autocorrelation function as the AR(p) process W_t. In Problem 7.3 we will see that scaling does not change the correlation which means that Y_t will have the same autocorrelation as W_t. We could also show this by multiplying Y_t with Y_{t-h} and take the expectation. Then we would get the same equation for autocovariance as for the AR(p) process.

Problem 7.3

a)

$$\begin{aligned}E(Z_t^2|Z_{t-1}^2, Z_{t-2}^2, ...) = E(h_t e_t^2|Z_{t-1}^2, Z_{t-2}^2, ...) = E(\alpha_0 + \sum_{i=1}^{p} \alpha_i Z_{t-1}^2 + \sum_{j=1}^{p} \beta_j h_{t-1}) \\ = \alpha_0 + \sum_{i=1}^{p} \alpha_i E(Z_{t-1}^2) + \sum_{j=1}^{p} \beta_j E(h_{t-1}) = \alpha_0 + \sum_{i=1}^{p} \alpha_i Z_{t-1}^2 + \sum_{j=1}^{p} \beta_j h_{t-1} \\ = h_t\end{aligned}$$

where we have used that $E(e_t^2) = 1$.

b) By inserting $U_t = Z_t^2 - h_t$ into the expression for Z_t^2 and solving for h_t, we end up with the equation for h_t.

c) In this exercise there is an error in the book. It should be $V_t = \alpha_1 U_{t-1}$. Using this, $\alpha_j^* = \alpha_{j+1}/\alpha_1$ and $U_t = Z_t^2 - h_t$ in the expression for h_t we end up with the initial equation for h_t.

Problem 11.3

We compute the third order cumulant function of the process in (11.3.1)

$$\begin{aligned}C_3(r, s) = E(X_t X_{t+r} X_{t+s}) = E(\sum_{i=0}^{\infty} \psi_i Z_{t-i} \sum_{j=0}^{\infty} \psi_j Z_{t+r-j} \sum_{k=0}^{\infty} \psi_k Z_{t+s-k}) \\ = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \psi_k \psi_j \psi_i E(Z_{t-i} Z_{t+r-j} Z_{t+s-k}) = \eta \sum_{i=0}^{\infty} \psi_i \psi_{i+r} \psi_{i+s} \\ = \eta \sum_{i=-\infty}^{\infty} \psi_i \psi_{i+r} \psi_{i+s}\end{aligned}$$

where $\psi_i = 0$ if $i < 0$.

Establishing (11.3.4) is outside the curriculum of the course, so we skip this.