TMA4285 Time series models
Solution to exercise 4, autumn 2018

October 2, 2018

Problem 3.1b

We write the ARMA processes as $\phi(B)X_t = \theta(B)Z_t$. The process $\{X_t : t \in Z\}$ is causal if and only if $\phi(z) \neq 0$ for each $|z| \leq 1$ and invertible if and only if $\theta(z) \neq 0$ for each $|z| \leq 1$.

$\phi(z) = 1 + 1.9z + 0.88z^2 = 0$ is solved by $z_1 = -5/4$ and $-10/11$. Hence $\{X_t : t \in Z\}$ is not causal.

$\theta(z) = 1 + 0.2z + 0.7z^2 = 0$ is solved by $z_1 = -(1 - i\sqrt{69})/7$ and $z_2 = -(1 + i\sqrt{69})/7$. Since $|z_1| = |z_2| = 70/7 > 1$, $\{X_t : t \in Z\}$ is invertible.

Problem 3.4

We have $X_t = 0.8X_{t-2} + Z_t$, where $\{Z_t : t \in Z\} \sim WN(0, \sigma^2)$. We multiply each side by X_{t-k} and take expected value. Then we get

$$\mathrm{E}[X_t, X_{t-k}] = 0.8\mathrm{E}[X_{t-2}, X_{t-k}] + \mathrm{E}[Z_t, X_{t-k}],$$

which gives us

$$\gamma(0) = 0.8\gamma(2) + \sigma^2$$
$$\gamma(k) = 0.8\gamma(k-2)$$

We use that $\gamma(k) = \gamma(-k)$ and need to solve

$$\begin{aligned}\gamma(0) - 0.8\gamma(2) &= \sigma^2 \\ \gamma(1) - 0.8\gamma(1) &= 0 \\ \gamma(2) - 0.8\gamma(0) &= 0\end{aligned}$$

First we see that $\gamma(1) = 0$ and therefore $\gamma(k) = 0$ if k is odd. Next we solve for $\gamma(0)$ and we get $\gamma(0) = \sigma^2/(1 - 0.82)$. It follows that $\gamma(2) = \gamma(0)0.8$, and that $\gamma(4) = \gamma(2)0.8 = \gamma(0)0.8^2$ and hence the ACF is

$$\rho(k) \begin{cases} 1, & k = 0 \\ 0.8^{k/2}, & k \text{ even} \\ 0, & \text{else} \end{cases}$$

The PACF can be computed as $\alpha(0) = 1$, $\alpha(h) = \phi_{hh}$ where ϕ_{hh} comes from that the best linear predictor of X_{h+1} has the form

$$\hat{X}_{h+1} = \sum_{i=1}^{h} \phi_{hi} X_{h+1-i}.$$

For an AR(2) process we have $\hat{X}_{h+1} = \phi X_h + \phi_2 X_{h-1}$ where we can identify $\alpha(0) = 1$, $\alpha(1) = 0$, $\alpha(2) = 0.8$ and $\alpha(h) = 0$ for $h \geq 3$.

Problem 3.8

We show that $\{W_t : t \in \mathbb{Z}\}$ is $\mathrm{WN}(0, \sigma_w^2)$.

$$\mathrm{E}(W_t) = \mathrm{E}(X_t - \frac{1}{\phi} X_{t-1}) = 0$$

we compute the ACVF.

$$\gamma_W(h) = \gamma_X(h) - \frac{1}{\phi}\gamma_X(h+1) - \frac{1}{\phi}\gamma_X(h-1) + \frac{1}{\phi^2}\gamma_X(h)$$

We use that the AR(1) series $\{X_t\}$ has ACVF $\gamma_X(h) = \frac{\sigma^2 \phi^h}{1-\phi^2}, h \geq 0$. If $h = 0$

$$\gamma_W(0) = \frac{\sigma^2}{1-\phi^2}(1 - \frac{\phi}{\phi} - \frac{\phi}{\phi} + \frac{1}{\phi^2}) = \frac{\sigma^2}{1-\phi^2}(1 - \frac{1}{\phi^2}) = \frac{\sigma^2}{\phi^2}.$$

For $h \geq 1$, we get $\gamma_W(h) = 0$. Hence $\{W_t : t \in \mathbb{Z}\}$ is $\mathrm{WN}(0, \sigma_w^2)$ with $\sigma_w^2 = \sigma^2/\phi^2$.

Problem 3.12

For an MA(1) process

$$\rho(h) = \begin{cases} 1, h = 0 \\ \theta/(1+\theta^2), |h| = 1, \\ 0, \text{ else} \end{cases}$$

Let $\alpha = \theta/(1+\theta^2)$. The system $R_n \phi_n = \rho_n$ becomes

$$\begin{pmatrix} 1 & \alpha & 0 & 0 & 0 & \dots & 0 \\ \alpha & 1 & \alpha & 0 & 0 & \dots & 0 \\ 0 & \alpha & 1 & \alpha & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha & 1 \end{pmatrix} \begin{pmatrix} \phi_{n,1} \\ \phi_{n,2} \\ \vdots \\ \phi_{n,n-1} \\ \phi_{n,n} \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

By solving this system using Gauss elimination and replacing α by $\theta/(1+\theta^2)$, we get $\phi_{nn} = -(-\theta)^n/(1+\theta^2+\cdots+\theta^{2n})$.

For an AR(2) process we have $\hat{X}_{h+1} = \phi_1 X_h + \phi_2 X_{h-1}$ where we can identify $\alpha(0) = 1$, $\alpha(1) = 0$, $\alpha(2) = 0.8$ and $\alpha(h) = 0$ for $h \geq 3$.

Problem 3.6. The ACVF for $\{X_t : t \in \mathbb{Z}\}$ is

$$\begin{aligned}\gamma_X(t+h,t) &= \mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}(Z_{t+h} + \theta Z_{t+h-1}, Z_t + \theta Z_{t-1}) \\ &= \gamma_Z(h) + \theta\gamma_Z(h+1) + \theta\gamma_Z(h-1) + \theta^2\gamma_Z(h) \\ &= \begin{cases} \sigma^2(1+\theta^2), & h = 0 \\ \sigma^2\theta, & |h| = 1. \end{cases}\end{aligned}$$

On the other hand, the ACVF for $\{Y_t : t \in \mathbb{Z}\}$ is

$$\begin{aligned}\gamma_Y(t+h,t) &= \mathrm{Cov}(Y_{t+h}, Y_t) = \mathrm{Cov}(\tilde{Z}_{t+h} + \theta^{-1}\tilde{Z}_{t+h-1}, \tilde{Z}_t + \theta^{-1}\tilde{Z}_{t-1}) \\ &= \gamma_{\tilde{Z}}(h) + \theta^{-1}\gamma_{\tilde{Z}}(h+1) + \theta^{-1}\gamma_{\tilde{Z}}(h-1) + \theta^{-2}\gamma_{\tilde{Z}}(h) \\ &= \begin{cases} \sigma^2\theta^2(1+\theta^{-2}) = \sigma^2(1+\theta^2), & h = 0 \\ \sigma^2\theta^2\theta^{-1} = \sigma^2\theta, & |h| = 1. \end{cases}\end{aligned}$$

Hence they are equal.

Problem 3.7. First we show that $\{W_t : t \in \mathbb{Z}\}$ is $\mathrm{WN}\left(0, \sigma_w^2\right)$.

$$\mathbb{E}[W_t] = \mathbb{E}\left[\sum_{j=0}^{\infty}(-\theta)^{-j}X_{t-j}\right] = \sum_{j=0}^{\infty}(-\theta)^{-j}\mathbb{E}[X_{t-j}] = 0,$$

since $\mathbb{E}[X_{t-j}] = 0$ for each j. Next we compute the ACVF of $\{W_t : t \in \mathbb{Z}\}$ for $h \geq 0$.

$$\begin{aligned}\gamma_W(t+h,t) &= \mathbb{E}[W_{t+h}W_t] = \mathbb{E}\left[\sum_{j=0}^{\infty}(-\theta)^{-j}X_{t+h-j}\sum_{k=0}^{\infty}(-\theta)^{-k}X_{t-k}\right] \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-\theta)^{-j}(-\theta)^{-k}\mathbb{E}[X_{t+h-j}X_{t-k}] = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-\theta)^{-j}(-\theta)^{-k}\gamma_X(h-j+k) \\ &= \left\{\gamma_X(r) = \sigma^2(1+\theta^2)\mathbf{1}_{\{0\}}(r) + \sigma^2\theta\mathbf{1}_{\{1\}}(|r|)\right\} \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-\theta)^{-(j+k)}\left(\sigma^2(1+\theta^2)\mathbf{1}_{\{j-k\}}(h) + \sigma^2\theta\mathbf{1}_{\{j-k+1\}}(h) + \sigma^2\theta\mathbf{1}_{\{j-k-1\}}(h)\right) \\ &= \sum_{j=h}^{\infty}(-\theta)^{-(j+j-h)}\sigma^2(1+\theta^2) + \sum_{j=h-1,j\geq 0}^{\infty}(-\theta)^{-(j+j-h+1)}\sigma^2\theta \\ &\quad + \sum_{j=h+1}^{\infty}(-\theta)^{-(j+j-h-1)}\sigma^2\theta \\ &= \sigma^2(1+\theta^2)(-\theta)^{-h}\sum_{j=h}^{\infty}(-\theta)^{-2(j-h)} + \sigma^2\theta(-\theta)^{-(h-1)}\sum_{j=h-1,j\geq 0}^{\infty}(-\theta)^{-2(j-(h-1))} \\ &\quad + \sigma^2\theta(-\theta)^{-(h+1)}\sum_{j=h+1}^{\infty}(-\theta)^{-2(j-(h+1))} \\ &= \sigma^2(1+\theta^2)(-\theta)^{-h}\frac{\theta^2}{\theta^2-1} + \sigma^2\theta(-\theta)^{-(h-1)}\frac{\theta^2}{\theta^2-1} + \sigma^2\theta^2\mathbf{1}_{\{0\}}(h) \\ &\quad + \sigma^2\theta(-\theta)^{-(h+1)}\frac{\theta^2}{\theta^2-1} \\ &= \sigma^2(-\theta)^{-h}\frac{\theta^2}{\theta^2-1}\left(1+\theta^2-\theta^2-1\right) + \sigma^2\theta^2\mathbf{1}_{\{0\}}(h) \\ &= \sigma^2\theta^2\mathbf{1}_{\{0\}}(h)\end{aligned}$$

Hence, $\{W_t : t \in \mathbb{Z}\}$ is WN $\left(0, \sigma_w^2\right)$ with $\sigma_w^2 = \sigma^2\theta^2$. To continue we have that

$$W_t = \sum_{j=0}^{\infty} (-\theta)^{-j} X_{t-j} = \sum_{j=0}^{\infty} \pi_j X_{t-j},$$

with $\pi_j = (-\theta)^{-j}$ and $\sum_{j=0}^{\infty} |\pi_j| = \sum_{j=0}^{\infty} \theta^{-j} < \infty$ so $\{X_t : t \in \mathbb{Z}\}$ is invertible and solves $\phi(B)X_t = \theta(B)W_t$ with $\pi(z) = \sum_{j=0}^{\infty} \pi_j z^j = \phi(z)/\theta(z)$. This implies that we must have

$$\sum_{j=0}^{\infty} \pi_j z^j = \sum_{j=0}^{\infty} \left(-\frac{z}{\theta}\right)^j = \frac{1}{1 + z/\theta} = \frac{\phi(z)}{\theta(z)}.$$

Hence, $\phi(z) = 1$ and $\theta(z) = 1 + z/\theta$, i.e. $\{X_t : t \in \mathbb{Z}\}$ satisfies $X_t = W_t + \theta^{-1} W_{t-1}$.

Problem 3.11. The PACF can be computed as $\alpha(0) = 1$, $\alpha(h) = \phi_{hh}$ where ϕ_{hh} comes from that the best linear predictor of X_{h+1} has the form

$$\hat{X}_{h+1} = \sum_{i=1}^{h} \phi_{hi} X_{h+1-i}.$$

In particular $\alpha(2) = \phi_{22}$ in the expression

$$\hat{X}_3 = \phi_{21} X_2 + \phi_{22} X_1.$$

The best linear predictor satisfies

$$\mathrm{Cov}(X_3 - \hat{X}_3, X_i) = 0, \qquad i = 1, 2.$$

This gives us

$$\begin{aligned}
&\mathrm{Cov}(X_3 - \hat{X}_3, X_1) = \mathrm{Cov}(X_3 - \phi_{21} X_2 - \phi_{22} X_1, X_1) \\
&\quad = \mathrm{Cov}(X_3, X_1) - \phi_{21} \mathrm{Cov}(X_2, X_1) - \phi_{22} \mathrm{Cov}(X_1, X_1) \\
&\quad = \gamma(2) - \phi_{21}\gamma(1) - \phi_{22}\gamma(0) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Cov}(X_3 - \hat{X}_3, X_2) = \mathrm{Cov}(X_3 - \phi_{21} X_2 - \phi_{22} X_1, X_2) \\
&\quad = \gamma(1) - \phi_{21}\gamma(0) - \phi_{22}\gamma(1) = 0.
\end{aligned}$$

Since we have an MA(1) process it has ACVF

$$\gamma(h) = \begin{cases} \sigma^2(1+\theta^2), & h = 0, \\ \sigma^2\theta, & |h| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, we have to solve the equations

$$\begin{aligned}
\phi_{21}\gamma(1) + \phi_{22}\gamma(0) &= 0 \\
(1 - \phi_{22})\gamma(1) - \phi_{21}\gamma(0) &= 0.
\end{aligned}$$

Solving this system of equations we find

$$\phi_{22} = -\frac{\theta^2}{\theta^4 + \theta^2 + 1}.$$

GT Exercises

Exercise 4

a The assumption that the process X is, which is an ARMA(2,1) process,

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t + \theta_t Z_{t-1} \tag{1}$$

is causal means that there exists a sequence of constants $\{\psi_t\}$ such that $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \qquad \text{for all t}$$

It also means that $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 \neq 0$ for all z such that $|z| \leq 1$. Similarly, the assumption of invertibility means that there exist constants $\{\pi_t\}$ such that $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} \qquad \text{for all t}$$

It also means that $\theta(z) = 1 + \theta_1 z \neq 0$ for all z such that $|z| \leq 1$.

b For an ARMA process the general expression of the sequence $\{\psi_t\}$ such that

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \qquad \text{for all t}$$

is determined by $\psi(z) = \frac{\theta(z)}{\phi(z)}$, which for an ARMA(2,1) process means

$$(1 - \phi_1 z_1 - \phi_2 z^2)(\psi_0 + \psi_1 z + \ldots) = 1 + \theta_1 z$$

Thus,

$$1 = \psi_0$$
$$\theta_1 = -\phi_1 \psi_0 + \psi_1 \implies \psi_1 = \theta_1 + \phi_1$$
$$0 = \theta_2 = \psi_2 - (\theta_1 + \phi_1)\phi_1 - \phi_2 \implies \psi_2 = \theta_2 + (\theta_1 + \phi_1)\phi_1 + \phi_2$$

For $j \geq 2$,

$$\psi_j = \phi_1 \psi_{j-1} + \phi_2 \psi_{j-2}$$

a linear combination of the two previous elements in the sequence $\{\psi_j\}$

c If eq. (1) is multiplied on each side by X_{t-k} and expectation is taken on both sides, then

$$E(X_t X_{t-k}) - \phi_1 E(X_{t-1} Xt - k) - \phi_2 E(X_{t-2} X_{t-k}) = E(Z_t X_{t-k}) + \theta_t E(Z_{t-1} X_{t-k})$$

$$\gamma(k) - \phi_1 \gamma(k-1) - \phi_2 \gamma(k-2) = E(\sum_{j=0}^{\infty} \psi_j Z_t Z_{t-k-j}) + \theta_t E(\sum_{j=0}^{\infty} \psi_j Z_{t-1} Z_{t-k-j})$$

which becomes

$$\gamma(k) - \phi_1 \gamma(k-1) - \phi_2 \gamma(k-2) = \sigma^2 \sum_{j=0}^{2} \theta_j \psi_{j-k} \qquad 0 \leq k \leq 1$$

and

$$\gamma(k) - \phi_1 \gamma(k-1) - \phi_2 \gamma(k-2) = 0 \qquad k \geq 2$$

Based on it,

$$\begin{aligned}
\gamma(0) - \phi_1 \gamma(1) - \phi_2 \gamma(2) &= \sigma^2 (1 + \theta_1 \psi_1) \\
\gamma(1) - \phi_1 \gamma(0) - \phi_2 \gamma(1) &= \sigma^2 \theta_1 \\
\gamma(2) - \phi_1 \gamma(1) - \phi_2 \gamma(0) &= 0
\end{aligned}$$

Thus,

$$\gamma(0) = \frac{1}{\phi_1} [(1 - \phi_2) \gamma(1) - \sigma^2 \theta_1]$$

with

$$\gamma(1) = \sigma^2 \left[\frac{(1 - \phi_2^2)\theta_1 + \phi_1 (1 + \theta_1 \psi_1)}{(1 - \phi_2^2)(1 - \phi_2) - \phi_1^2 (1 + \phi_2)} \right]$$

Based on $\gamma(1)$ and $\gamma(0)$ we can compute:

$$\gamma(k) = \phi_1 \gamma(k-1) + \phi_2 \gamma(k-2)$$

for all $k \geq 2$.

From this recurrence $\rho(k)$ can be computed for all k.

d For the zero-mean process $\{X_t\}$ with $E(X_i X_j) = \kappa(i,j)$, the innovation algorithm states the one step predictors $\hat{X}_{n+1}$ are given by

$$\hat{X}_{n+1} = \begin{cases} 0 & \text{if } n = 0 \\ \sum_{j=1}^{n} \theta_{n_j}(X_{n+1-j} - \hat{X}_{n+1-j}) & \text{if } n \geq 1 \end{cases}$$

where

$$\begin{cases} \nu_0 & = \kappa(1,1) \\ \theta_{n,n-k} & = \nu_k^{-1}\big(\kappa(n+1,k+1) - \sum_{j=0}^{k-1} \theta_{k,k-j}\theta_{n,n-j}\nu_j\big), \quad k = 0,1,\ldots,n-1 \\ \nu_n & = \kappa(n+1,n+1) - \sum_{j=0}^{n-1} \theta_{n,n-j}^2 \nu_j \end{cases}$$

- When n=0, $\hat{X}_{n+1} = \hat{X}_1 = 0$
- For $n = 1$ we have:

  $$\hat{X}_2 = \theta_{11} X_1 = \frac{\gamma(1)}{\gamma(0)} X_1$$

  since

  $$\begin{aligned} \theta_{11} &= \nu_0^{-1}\kappa(1,0) \\ &= \frac{\gamma(1)}{\gamma(0)} \end{aligned}$$

  which can be computed making use of the recursion in part c.
- When $n = 2$:

  $$\begin{aligned} \hat{X}_3 &= \theta_{21}(X_2 - \hat{X}_2) + \theta_{22}(X_1 - \hat{X}_1) \\ &= \theta_{21}\left(X_2 - \frac{\gamma(1)}{\gamma(0)} X_1\right) + \theta_{22}(X_1) \end{aligned}$$

θ_{21} and θ_{22} are obtained as follows:

$$\begin{aligned}
\nu_1 &= \kappa(2,2) - \theta_{11}^2\nu_0 \\
&= (\gamma(0) - \theta_{11}^2\gamma(0)) \\
&= \gamma(0) - \frac{\gamma^2(1)}{\gamma(0)} \\
&= \frac{\gamma^2(0) - \gamma^2(1)}{\gamma(0)}
\end{aligned}$$

Then,

$$\begin{aligned}
\theta_{22} &= \frac{\gamma(2)}{\nu_0} \\
&= \frac{\gamma(2)}{\gamma(0)}
\end{aligned}$$

$$\begin{aligned}
\theta_{21} &= \nu_1^{-1}[\gamma(1) - \theta_{22}\theta_{11}\nu_0] \\
&= \frac{\gamma(1) - \frac{\gamma(2)\gamma(1)}{\gamma(0)}}{\frac{\gamma^2(0)-\gamma^2(1)}{\gamma(0)}} \\
&= \frac{\gamma(1)[\gamma(0) - \gamma(2)]}{\gamma^2(0) - \gamma^2(1)}
\end{aligned}$$

– For $n = 3$,

$$\hat{X}_4 = \theta_{31}(X_3 - \hat{X}_3) + \theta_{32}(X_2 - \hat{X}_2) + \theta_{33}(X_1 - \hat{X}_1)$$

First, we need to compute ν_2

$$\begin{aligned}
\nu_2 &= \kappa(3,3) - \theta_{22}^2\nu_0 - theta_{21}^2\nu_1 \\
&= \gamma(0)(1 - \theta_{22}^2) - \theta_{21}^2
\end{aligned}$$

Then,

$$\theta_{33} = \frac{\gamma(3)}{\gamma(0)},$$

$$\theta_{32} = \frac{\gamma(2) - \theta_{11}\theta_{33}\gamma(0)}{\nu_1}$$

and

$$\theta_{31} = \frac{\gamma(1) - \theta_{22}\theta_{33}\gamma(0) - \theta_{21}\theta_{32}\nu_1}{\nu_2}$$

The general expression of the Durbin-Levinson algorithm is

$$\hat{X}_{n+1} = \phi_{n1}X_n + \cdots + \phi_{nn}X_1 \text{for } n \geq 1$$

where $\phi_{11} = \gamma(1)/\gamma(0)$, $\nu_0 = \gamma(0)$,

$$\phi_{nn} = \left[\gamma(n) - \sum_{j=1}^{n-1} \phi_{n-1,j}\gamma(n-j)\right]\nu_{n-1}^{-1}$$

$$\begin{bmatrix} \phi_{n1} \\ \vdots \\ \phi_{n,n-1} \end{bmatrix} = \begin{bmatrix} \phi_{n-1,1} \\ \vdots \\ \phi_{n-1,n-1} \end{bmatrix} - \phi_{nn} \begin{bmatrix} \phi_{n-1,n-1} \\ \vdots \\ \phi_{n-1,1} \end{bmatrix}$$

and

$$\nu_n = \nu_{n-1}[1 - \phi_{nn}^2]$$

- $n = 1$

$$\begin{aligned} \hat{X}_2 &= \phi_{11}X_1 \\ &= \frac{\gamma(1)}{\gamma(0)}X_1 \end{aligned}$$

  since $\phi_{11} = \frac{\gamma(1)}{\gamma(0)}$

- $n = 2$

$$\hat{X}_3 = \phi_{21}X_2 + \phi_{22}X_1$$

where

$\phi_{22} = \big[\gamma(2) - \phi_{11}\gamma(0)\big]\nu_1^{-1}$

with $\nu_1 = \nu_0[1-\phi_{11}^2] = \frac{\gamma^2(0)-\gamma^2(1)}{\gamma(0)}$. Then,

$$\phi_{22} = \frac{\gamma(0)\gamma(2) - \gamma^2(1)}{\gamma^2(0) - \gamma^2(1)}$$

and

$$\begin{aligned}\phi_{21} &= \phi_{11}(1-\phi_{22}) \\ &= \frac{\gamma(1)}{\gamma(0)}\left[1 - \frac{\gamma(0)\gamma(2) - \gamma^2(1)}{\gamma^2(0) - \gamma^2(1)}\right]\end{aligned}$$

– $n = 3$

$$\hat{X}_4 = \phi_{31}X_3 + \phi_{32}X_2 + \phi_{33}X_1$$

with

$$\phi_{33} = \left[\gamma(3) - \phi_{21}\gamma(2) - \phi_{22}\gamma(1)\right]\nu_2^{-1}$$

where $\nu_2 = \nu_1[1-\phi_{22}^2]$,

$$\phi_{32} = \phi_{22} - \phi_{33}\phi_{21}$$

and

$$\phi_{31} = \phi_{21} - \phi_{33}\phi_{22}$$

Both the innovations algorithm and Durbin-Watson algorithm are a recursive way of performing prediction. They also work as a preliminary estimation of ARMA(p,q) processes. Durbin-Watson algorithm is also useful as a way to compute the Partial Autocorrelation Function as seen in part e.

e Based on chapter 8 from Brockwell, Davids (1991),

$$\alpha(k) = \phi_{kk}, \quad k \geq 1$$

Then,

$$\alpha_1 = \phi_{11} = \frac{\gamma(1)}{\gamma(0)}$$

$$\alpha_2 = \phi_{22} = \frac{\gamma(0)\gamma(2) - \gamma^2(1)}{\gamma^2(0) - \gamma^2(1)}$$

$$\alpha_3 = \phi_{33} = \Big[\gamma(3) - \phi_{21}\gamma(2) - \phi_{22}\gamma(1)\Big]\nu_2^{-1}$$

and in general

$$\alpha_k = \phi_{kk} = \Big[\gamma(k) - \sum_{j=1}^{k-1}\phi_{k-1,j}\gamma(k-j)\Big]\nu_{k-1}^{-1}$$

f For this case, the model parameter Θ is

$$\Theta = (\phi_1, \phi_2, \theta_1, \sigma^2)$$

The parameter space depends on the causality and invertibility of the ARMA(2,1) process.

- $\Omega_{\sigma^2} = [0, \infty)$.
- The invertibility of the ARMA(2,1) process means $1 + \theta_1 z \neq 0$ for all $|z| \leq 1$. Then,

  $$\theta_1 \neq -\frac{1}{z} \quad \text{for all } |z| \leq 1$$

  which implies

  $$\Omega_{\theta_1} = (-1, 1)$$

- For the parameters ϕ_1 and ϕ_2, the causality condition means $1 - \phi_1 z - \phi_2 z^2 \neq 0$ for all $|z| \leq 1$. Thus, considering the solutions:

  $$z_1 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}$$
  $$z_2 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2},$$

they must satisfy $|z_1| > 1$ and $|z_2| > 1$. That is:

$$\left|\frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}\right| > 1 \quad \text{and}$$

$$\left|\frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}\right| > 1$$

$$\text{Thus, } \Omega_{(\phi_1,\phi_2)} = \left\{(\phi_1, \phi_2) : \left|\tfrac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}\right| > 1; \left|\tfrac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}\right| > 1\right\}$$

So,

$$\Omega_\Theta = \Omega_{(\phi_1,\phi_2)} \times \Omega_{\theta_1} \times \Omega_{\sigma^2}$$

g Based on the commutative diagram of statistics we can affirm that the covariance function $\gamma(h)$ is a parameter since it is unkown and, from part c, $\gamma(0)$, $\gamma(1)$ and $\gamma(k); k \geq 2$ can all be expressed as functions of $\Theta = (\phi_1, \phi_2, \theta_1, \sigma^2)$.