TMA4285 Time series models
Solution to exercise 9, autumn 2018

November 14, 2018

Problem 9.1

We begin with equation (9.1.11) and insert recursively

$$\begin{aligned}\mathbf{X}_t &= F\mathbf{X}_{t-1} + \mathbf{V}_{t-1} = F(F\mathbf{X}_{t-2} + \mathbf{V}_{t-2}) + \mathbf{V}_{t-1} = F^2(F\mathbf{X}_{t-3} + \mathbf{V}_{t-3}) + F\mathbf{V}_{t-2} + \mathbf{V}_{t-1} \\ &= \cdots = \sum_{j=0}^{\infty} F^j \mathbf{V}_{t-1-j}\end{aligned}$$

The condition $F^k \to 0$ as $k \to \infty$ ensures convergence of the infinite series.

Next, we want to deduce that $\{(\mathbf{X}_t^T, \mathbf{Y}_t^T)^T\}$ is a multivariate stationary process. What is needed to deduce this is the vector version of proposition 2.2.1.

Problem 9.3

We want to show that $det(zI - F) = z^p\phi(z^{-1})$, and we can do this using induction.

First, we show that $det(zI - F) = z^p\phi(z^{-1})$ for $p = 1$

$$det(z - \phi_1) = z - \phi_1 = z(1 - \frac{\phi_1}{z}) = z\phi(z^{-1})$$

Next we assume that $det(zI - F) = z^p\phi(z^{-1})$ holds for $p = k$ and show that then it also holds for $p = k + 1$.

$$det(zI - F) = z^{k+1}\phi(z^{-1})$$

$$det\begin{pmatrix} z & -1 & 0 & \dots & 0 \\ 0 & z & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & z & -1 \\ \phi_{k+1} & \phi_k & \dots & \phi_2 & z - \phi_1 \end{pmatrix} = z^{k+1}(1 - \frac{\phi_1}{z} - \dots - \frac{\phi_{k+1}}{z^{k+1}})$$

$$z{\cdot}det\begin{pmatrix} z & -1 & \dots & 0 & \\ & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & z & -1 & \\ \phi_k & \dots & \phi_2 & z - \phi_1 & \end{pmatrix} \pm \phi_{k+1} \cdot det\begin{pmatrix} -1 & 0 & \dots & 0 \\ z & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & z & -1 \end{pmatrix} = z^{k+1}(1 - \frac{\phi_1}{z} - \dots - \frac{\phi_k}{z^k}) - \phi_{k+1}$$

$$z{\cdot}det\begin{pmatrix} z & -1 & \dots & 0 & \\ & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & z & -1 & \\ \phi_k & \dots & \phi_2 & z - \phi_1 & \end{pmatrix} \pm \phi_{k+1}(\mp 1) = z^{k+1}(1 - \frac{\phi_1}{z} - \dots - \frac{\phi_k}{z^k}) - \phi_{k+1}$$

$$det\begin{pmatrix} z & -1 & \dots & 0 & \\ & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & z & -1 & \\ \phi_k & \dots & \phi_2 & z - \phi_1 & \end{pmatrix} = z^k(1 - \frac{\phi_1}{z} - \dots - \frac{\phi_k}{z^k})$$

$$det(zI - F) = z^k\phi(z^{-1}),$$

which was our assumption.

Problem 9.11

a) We want to show that if $XS = B$ can be solved for X, then $X = BS^{-1}$ is a solution for any generalized inverse S^{-1} of S.

In short, we insert the solution into the equation and verify that both sides are equal

$$XS = B \rightarrow BS^{-1}S = B \rightarrow B = B$$

b) Let $P(\mathbf{X}|\mathbf{Y}) = \hat{E}(\mathbf{X}|\mathbf{Y}) = M\mathbf{Y}$, where $\mathbf{X}$ and $\mathbf{Y}$ are vectors of dimension v and w, and M is a matrix of dimension $v \times w$. To find the best linear prediction, we must minimize the mean squared errors $E((\mathbf{X} - M\mathbf{Y})^2)$. This corresponds to solving

$$\begin{aligned} E((\mathbf{X} - M\mathbf{Y})\mathbf{Y}^T) &= 0 \\ &\downarrow \\ E(\mathbf{X}\mathbf{Y}^T) &= ME(\mathbf{Y}\mathbf{Y}^T) \\ &\downarrow \\ M &= E(\mathbf{X}\mathbf{Y}^T)[E(\mathbf{Y}\mathbf{Y}^T)]^{-1}, \end{aligned}$$

where $[E(\mathbf{Y}\mathbf{Y}^T)]^{-1}$ is any generalized inverse of $E(\mathbf{Y}\mathbf{Y}^T)$.

Problem 9.12

What we need to do is show the relation in equation (9.4.3). Let $\mathcal{H}_t$ be the vector space consisting of all linear combinations of $\mathbf{Y}_0, ..., \mathbf{Y}_t$,

$$\mathcal{H}_t = \{\sum_{i=0}^{t} c_i \mathbf{Y}_i | c_i \text{ matrix } \}$$

Furthermore, let $\mathcal{H}_{t-1}$ be the vector space consisting of all linear combinations of $\mathbf{Y}_0, ..., \mathbf{Y}_{t-1}$. That is

$$\mathcal{H}_{t-1} = \{\sum_{i=0}^{t-1} c_i \mathbf{Y}_i | c_i \text{ matrix } \}$$

Next, define the innovations in the same way as in the book, $\mathbf{I}_t = \mathbf{Y}_t - P_{t-1}\mathbf{Y}_t$. Here, $\mathbf{Y}_t$ is a vector in $\mathcal{H}_t$ and $P_{t-1}\mathbf{Y}_t$ is the projection onto $\mathcal{H}_{t-1}$. The innovation $\mathbf{I}_t$ is a linear combination of $\mathbf{Y}_0, ..., \mathbf{Y}_t$ and is orthogonal to $\mathcal{H}_{t-1}$. This means that

$$\mathcal{H}_t = \mathcal{H}_{t-1} \oplus \mathcal{H}_{I_t}, \tag{1}$$

where $\mathcal{H}_{\mathbf{I}_t} = \{M\mathbf{I}_t\}$. A consequence of (1) is that

$$P_t = P_{t-1} + P_{I_t}$$

Problem 9.17

a) The two equations we will be using are

$$\hat{\mathbf{X}}_{t+1} = F_t\hat{\mathbf{X}}_t + \Theta_t\Delta_t^{-1}(\mathbf{Y}_t - G_t\hat{\mathbf{X}}_t) \tag{2}$$

and

$$P_t\mathbf{X}_t = P_{t-1}\mathbf{X}_t + \Omega_t G_t^T\Delta_t^{-1}(\mathbf{Y}_t - G_t\hat{\mathbf{X}}_t) \tag{3}$$

We note by definition $P_t\mathbf{X}_t = \mathbf{X}_{t|t}$ and $P_{t-1}\mathbf{X}_t = \hat{\mathbf{X}}_t$.

We can rewrite equation (3) as

$$\mathbf{Y}_t - G_t\hat{\mathbf{X}}_t = (\Delta_t^{-1}G_t^T\Omega_t)^{-1}(P_t\mathbf{X}_t - P_{t-1}\mathbf{X}_t)$$

and insert this into equation (2). This gives

$$\begin{aligned}\hat{\mathbf{X}}_{t+1} &= F_t\hat{\mathbf{X}}_t + \Theta_t\Delta_t^{-1}(\Delta_t^{-1}G_t^T\Omega_t)^{-1}(P_t\mathbf{X}_t - P_{t-1}\mathbf{X}_t)\\ &= F_t\hat{\mathbf{X}}_t + \Theta_t(G_t^T\Omega_t)^{-1}(P_t\mathbf{X}_t - P_{t-1}\mathbf{X}_t)\\ &= F_t\hat{\mathbf{X}}_t + F_t(\mathbf{X}_{t|t} - \hat{\mathbf{X}}_t)\\ &= F_t\mathbf{X}_{t|t}\end{aligned}$$

where we have used that $\Theta_t = F_t\Omega_t G_t^T$ from the Kalman prediction.

b) We now have

$$\hat{\mathbf{X}}_{t+1} = F_t\mathbf{X}_{t|t} \tag{4}$$

and

$$\mathbf{X}_{t|t} = \hat{\mathbf{X}}_t + \Omega_t G_t^T\Delta_t^{-1}(\mathbf{Y}_t - G_t\hat{\mathbf{X}}_t) \tag{5}$$

By inserting (4) into (5) we get

$$\mathbf{X}_{t|t} = F_{t-1}\mathbf{X}_{t-1|t-1} + \Omega_t G_t^T\Delta_t^{-1}(\mathbf{Y}_t - G_tF_{t-1}\mathbf{X}_{t-1|t-1})$$

For t=1, equation (5) gives

$$\mathbf{X}_{1|1} = \hat{\mathbf{X}}_1 + \Omega_1 G_1^T\Delta_1^{-1}(\mathbf{Y}_1 - G_1\hat{\mathbf{X}}_1)$$

Problem 8.9. Let $\mathbf{Y}_t$ consist of $\mathbf{Y}_{t,1}$ and $\mathbf{Y}_{t,2}$, then we can write

$$\begin{aligned}\mathbf{Y}_t &= \begin{bmatrix} \mathbf{Y}_{t,1} \\ \mathbf{Y}_{t,1} \end{bmatrix} = \begin{bmatrix} G_1\mathbf{X}_{t,1} + \mathbf{W}_{t,1} \\ G_2\mathbf{X}_{t,2} + \mathbf{W}_{t,2} \end{bmatrix} = \begin{bmatrix} G_1\mathbf{X}_{t,1} \\ G_2\mathbf{X}_{t,2} \end{bmatrix} + \begin{bmatrix} \mathbf{W}_{t,1} \\ \mathbf{W}_{t,2} \end{bmatrix} \\ &= \begin{bmatrix} G_1 & \mathbf{0} \\ \mathbf{0} & G_2 \end{bmatrix} \begin{bmatrix} \mathbf{X}_{t,1} \\ \mathbf{X}_{t,2} \end{bmatrix} + \begin{bmatrix} \mathbf{W}_{t,1} \\ \mathbf{W}_{t,2} \end{bmatrix}.\end{aligned}$$

Set

$$G = \begin{bmatrix} G_1 & \mathbf{0} \\ \mathbf{0} & G_2 \end{bmatrix}, \quad \mathbf{X}_t = \begin{bmatrix} \mathbf{X}_{t,1} \\ \mathbf{X}_{t,1} \end{bmatrix} \quad \text{and} \quad \mathbf{W}_t = \begin{bmatrix} \mathbf{W}_{t,1} \\ \mathbf{W}_{t,2} \end{bmatrix}$$

then we have $\mathbf{Y}_t = G\mathbf{X}_t + \mathbf{W}_t$. Similarly we have that

$$\begin{aligned}\mathbf{X}_{t+1} &= \begin{bmatrix} \mathbf{X}_{t+1,1} \\ \mathbf{X}_{t+1,1} \end{bmatrix} = \begin{bmatrix} F_1\mathbf{X}_{t,1} + \mathbf{V}_{t,1} \\ F_2\mathbf{X}_{t,2} + \mathbf{V}_{t,2} \end{bmatrix} = \begin{bmatrix} F_1\mathbf{X}_{t,1} \\ F_2\mathbf{X}_{t,2} \end{bmatrix} + \begin{bmatrix} \mathbf{V}_{t,1} \\ \mathbf{V}_{t,2} \end{bmatrix} \\ &= \begin{bmatrix} F_1 & \mathbf{0} \\ \mathbf{0} & F_2 \end{bmatrix} \begin{bmatrix} \mathbf{X}_{t,1} \\ \mathbf{X}_{t,2} \end{bmatrix} + \begin{bmatrix} \mathbf{V}_{t,1} \\ \mathbf{V}_{t,2} \end{bmatrix}\end{aligned}$$

and set

$$F = \begin{bmatrix} F_1 & \mathbf{0} \\ \mathbf{0} & F_2 \end{bmatrix} \quad \text{and} \quad \mathbf{V}_t = \begin{bmatrix} \mathbf{V}_{t,1} \\ \mathbf{V}_{t,2} \end{bmatrix}.$$

Finally we have the state-space representation

$$\begin{aligned}\mathbf{Y}_t &= G\mathbf{X}_t + \mathbf{W}_t \\ \mathbf{X}_{t+1} &= F\mathbf{X}_t + \mathbf{V}_t.\end{aligned}$$

Problem 8.13. We have to solve

$$\Omega + \sigma_v^2 - \frac{\Omega^2}{\Omega + \sigma_w^2} = \Omega$$

which is equivalent to

$$\frac{\Omega^2}{\Omega + \sigma_w^2} - \sigma_v^2 = 0.$$

Multiplying with $\Omega + \sigma_w^2$ we get

$$\Omega^2 - \Omega\sigma_v^2 - \sigma_w^2\sigma_v^2 = 0,$$

which has the solutions

$$\Omega = \frac{1}{2}\sigma_v^2 \pm \sqrt{\frac{\sigma_v^4}{4} + \sigma_w^2\sigma_v^2} = \frac{\sigma_v^2 \pm \sqrt{\sigma_v^4 + 4\sigma_w^2\sigma_v^2}}{2}.$$

Since $\Omega \geq 0$ we have the positive root which is the solution we wanted.

Problem 8.14. We have that

$$\Omega_{t+1} = \Omega_t + \sigma_v^2 - \frac{\Omega_t^2}{\Omega_t + \sigma_w^2}$$

and since $\sigma_v^2 = \Omega^2/(\Omega + \sigma_w^2)$ substracting Ω yields

$$\begin{aligned}\Omega_{t+1} - \Omega &= \Omega_t + \frac{\Omega^2}{\Omega + \sigma_w^2} - \frac{\Omega_t^2}{\Omega_t + \sigma_w^2} - \Omega \\ &= \frac{\Omega_t\left(\Omega_t + \sigma_w^2\right) - \Omega_t^2}{\Omega_t + \sigma_w^2} - \frac{\Omega\left(\Omega + \sigma_w^2\right) - \Omega^2}{\Omega + \sigma_w^2} \\ &= \frac{\Omega_t\sigma_w^2}{\Omega_t + \sigma_w^2} - \frac{\Omega\sigma_w^2}{\Omega + \sigma_w^2} \\ &= \sigma_w^2\left(\frac{\Omega_t}{\Omega_t + \sigma_w^2} - \frac{\Omega}{\Omega + \sigma_w^2}\right).\end{aligned}$$