TMA4285 Time series models
Solution to exercise 10, autumn 2020

November 15, 2020

Problem 9.1

We begin with equation (9.1.11) and insert recursively

$$\begin{aligned}\mathbf{X}_t &= F\mathbf{X}_{t-1} + \mathbf{V}_{t-1} = F(F\mathbf{X}_{t-2} + \mathbf{V}_{t-2}) + \mathbf{V}_{t-1} = F^2(F\mathbf{X}_{t-3} + \mathbf{V}_{t-3}) + F\mathbf{V}_{t-2} + \mathbf{V}_{t-1} \\ &= \cdots = \sum_{j=0}^{\infty} F^j \mathbf{V}_{t-1-j}\end{aligned}$$

The condition $F^k \to 0$ as $k \to \infty$ ensures convergence of the infinite series.

Next, we want to deduce that $\{(\mathbf{X}_t^T, \mathbf{Y}_t^T)^T\}$ is a multivariate stationary process. What is needed to deduce this is the vector version of proposition 2.2.1.

Problem 9.3

We want to show that $det(zI - F) = z^p\phi(z^{-1})$, and we can do this using induction.
First, we show that $det(zI - F) = z^p\phi(z^{-1})$ for $p = 1$

$$det(z - \phi_1) = z - \phi_1 = z(1 - \frac{\phi_1}{z}) = z\phi(z^{-1})$$

Next we assume that $det(zI - F) = z^p\phi(z^{-1})$ holds for $p = k$ and show that then it also holds for $p = k + 1$.

$$det(zI - F) = z^{k+1}\phi(z^{-1})$$

$$det\begin{pmatrix} z & -1 & 0 & \dots & 0 \\ 0 & z & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & z & -1 \\ \phi_{k+1} & \phi_k & \dots & \phi_2 & z - \phi_1 \end{pmatrix} = z^{k+1}(1 - \frac{\phi_1}{z} - \dots - \frac{\phi_{k+1}}{z^{k+1}})$$

$$z{\cdot}det\begin{pmatrix} z & -1 & \dots & 0 & \\ & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & z & -1 & \\ \phi_k & \dots & \phi_2 & z - \phi_1 & \end{pmatrix} \pm \phi_{k+1} \cdot det\begin{pmatrix} -1 & 0 & \dots & 0 \\ z & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & z & -1 \end{pmatrix} = z^{k+1}(1 - \frac{\phi_1}{z} - \dots - \frac{\phi_k}{z^k}) - \phi_{k+1}$$

$$z{\cdot}det\begin{pmatrix} z & -1 & \dots & 0 & \\ & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & z & -1 & \\ \phi_k & \dots & \phi_2 & z - \phi_1 & \end{pmatrix} \pm \phi_{k+1}(\mp 1) = z^{k+1}(1 - \frac{\phi_1}{z} - \dots - \frac{\phi_k}{z^k}) - \phi_{k+1}$$

$$det\begin{pmatrix} z & -1 & \dots & 0 & \\ & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & z & -1 & \\ \phi_k & \dots & \phi_2 & z - \phi_1 & \end{pmatrix} = z^k(1 - \frac{\phi_1}{z} - \dots - \frac{\phi_k}{z^k})$$

$$det(zI - F) = z^k\phi(z^{-1}),$$

which was our assumption.

Problem 9.9

Let $\mathbf{X}_t = (\mathbf{X'}_{t1}, \mathbf{X'}_{t2})'$, $\mathbf{V}_t = (\mathbf{V'}_{t1}, \mathbf{V'}_{t2})'$ and $\mathbf{W}_t = (\mathbf{W'}_{t1}, \mathbf{W'}_{t2})'$. We can express $\mathbf{Y}_t$ as:

$$\mathbf{Y}_t = \mathbf{G}_t^* \cdot \mathbf{X}_t + \mathbf{W}_t$$

with:

$$\mathbf{G}_t^* = \begin{bmatrix} \mathbf{G}_1 & 0 \\ 0 & \mathbf{G}_2 \end{bmatrix}$$

while $\mathbf{X}_{t+1}$ can be expressed as:

$$\mathbf{X}_{t+1} = \mathbf{F}_t^* \cdot \mathbf{X}_t + \mathbf{V}_t$$

with:

$$\mathbf{F}_t^* = \begin{bmatrix} \mathbf{F}_1 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix}$$

Problem 9.11

a) We want to show that if $XS = B$ can be solved for X, then $X = BS^{-1}$ is a solution for any generalized inverse S^{-1} of S.

In short, we insert the solution into the equation and verify that both sides are equal

$$XS = B \to BS^{-1}S = B \to B = B$$

b) Let $P(\mathbf{X}|\mathbf{Y}) = \hat{E}(\mathbf{X}|\mathbf{Y}) = M\mathbf{Y}$, where $\mathbf{X}$ and $\mathbf{Y}$ are vectors of dimension v and w, and M is a matrix of dimension $v \times w$. To find the best linear prediction, we must minimize the mean squared errors $E((\mathbf{X} - M\mathbf{Y})^2)$. This corresponds to solving

$$\begin{aligned} E((\mathbf{X} - M\mathbf{Y})\mathbf{Y}^T) &= 0 \\ &\downarrow \\ E(\mathbf{X}\mathbf{Y}^T) &= ME(\mathbf{Y}\mathbf{Y}^T) \\ &\downarrow \\ M &= E(\mathbf{X}\mathbf{Y}^T)[E(\mathbf{Y}\mathbf{Y}^T)]^{-1}, \end{aligned}$$

where $[E(\mathbf{Y}\mathbf{Y}^T)]^{-1}$ is any generalized inverse of $E(\mathbf{Y}\mathbf{Y}^T)$.

Problem 9.12

What we need to do is show the relation in equation (9.4.3). Let $\mathcal{H}_t$ be the vector space consisting of all linear combinations of $\mathbf{Y}_0, ..., \mathbf{Y}_t$,

$$\mathcal{H}_t = \{\sum_{i=0}^{t} c_i \mathbf{Y}_i | c_i \text{ matrix } \}$$

Furthermore, let $\mathcal{H}_{t-1}$ be the vector space consisting of all linear combinations of $\mathbf{Y}_0, ..., \mathbf{Y}_{t-1}$. That is

$$\mathcal{H}_{t-1} = \{\sum_{i=0}^{t-1} c_i \mathbf{Y}_i | c_i \text{ matrix } \}$$

Next, define the innovations in the same way as in the book, $\mathbf{I}_t = \mathbf{Y}_t - P_{t-1}\mathbf{Y}_t$. Here, $\mathbf{Y}_t$ is a vector in $\mathcal{H}_t$ and $P_{t-1}\mathbf{Y}_t$ is the projection onto $\mathcal{H}_{t-1}$. The innovation $\mathbf{I}_t$ is a linear combination of $\mathbf{Y}_0, ..., \mathbf{Y}_t$ and is orthogonal to $\mathcal{H}_{t-1}$. This means that

$$\mathcal{H}_t = \mathcal{H}_{t-1} \oplus \mathcal{H}_{I_t}, \tag{1}$$

where $\mathcal{H}_{\mathbf{I}_t} = \{M\mathbf{I}_t\}$. A consequence of (1) is that

$$P_t = P_{t-1} + P_{I_t}$$

Problem 9.17

a) The two equations we will be using are

$$\hat{\mathbf{X}}_{t+1} = F_t\hat{\mathbf{X}}_t + \Theta_t\Delta_t^{-1}(\mathbf{Y}_t - G_t\hat{\mathbf{X}}_t) \tag{2}$$

and

$$P_t\mathbf{X}_t = P_{t-1}\mathbf{X}_t + \Omega_t G_t^T\Delta_t^{-1}(\mathbf{Y}_t - G_t\hat{\mathbf{X}}_t) \tag{3}$$

We note by definition $P_t\mathbf{X}_t = \mathbf{X}_{t|t}$ and $P_{t-1}\mathbf{X}_t = \hat{\mathbf{X}}_t$.

We can rewrite equation (3) as

$$\mathbf{Y}_t - G_t\hat{\mathbf{X}}_t = (\Delta_t^{-1}G_t^T\Omega_t)^{-1}(P_t\mathbf{X}_t - P_{t-1}\mathbf{X}_t)$$

and insert this into equation (2). This gives

$$\begin{aligned}
\hat{\mathbf{X}}_{t+1} &= F_t\hat{\mathbf{X}}_t + \Theta_t\Delta_t^{-1}(\Delta_t^{-1}G_t^T\Omega_t)^{-1}(P_t\mathbf{X}_t - P_{t-1}\mathbf{X}_t) \\
&= F_t\hat{\mathbf{X}}_t + \Theta_t(G_t^T\Omega_t)^{-1}(P_t\mathbf{X}_t - P_{t-1}\mathbf{X}_t) \\
&= F_t\hat{\mathbf{X}}_t + F_t(\mathbf{X}_{t|t} - \hat{\mathbf{X}}_t) \\
&= F_t\mathbf{X}_{t|t}
\end{aligned}$$

where we have used that $\Theta_t = F_t\Omega_t G_t^T$ from the Kalman prediction.

b) We now have

$$\hat{\mathbf{X}}_{t+1} = F_t\mathbf{X}_{t|t} \tag{4}$$

and

$$\mathbf{X}_{t|t} = \hat{\mathbf{X}}_t + \Omega_t G_t^T\Delta_t^{-1}(\mathbf{Y}_t - G_t\hat{\mathbf{X}}_t) \tag{5}$$

By inserting (4) into (5) we get

$$\mathbf{X}_{t|t} = F_{t-1}\mathbf{X}_{t-1|t-1} + \Omega_t G_t^T\Delta_t^{-1}(\mathbf{Y}_t - G_t F_{t-1}\mathbf{X}_{t-1|t-1})$$

For t=1, equation (5) gives

$$\mathbf{X}_{1|1} = \hat{\mathbf{X}}_1 + \Omega_1 G_1^T\Delta_1^{-1}(\mathbf{Y}_1 - G_1\hat{\mathbf{X}}_1)$$