TMA4285 Time series models
Solution to exercise 2, autumn 2020

September 12, 2020

Problem 2.9

a)

$$E(Y_t) = E(X_t) + E(W_t) = E(X_t)$$

To find $E(X_t)$, we must express X_t as $X_t = \psi(B)Z_t$.

$$\psi(B) = \frac{1}{1-\phi(B)} \iff (1-\phi B)(1+\psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots) = 1$$

By collecting terms with same power of B, we get

$$\begin{aligned} &\psi_1 - \phi = 1 \rightarrow \psi_1 = \phi \\ &\psi_2 - \phi\psi_1 = 0 \rightarrow \psi_2 = \phi^2 \\ &\psi_3 - \phi\psi_2 = 0 \rightarrow \psi_3 = \phi^3 \\ &\vdots \end{aligned}$$

Thus, $E(Y_t) = E(X_t) = E(Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2} + \dots) = 0.$

Next, we find the autocovariance function

$$\begin{aligned} Cov(Y_t, Y_{t+h}) &= Cov(X_t + W_t, X_{t+h} + W_{t+h}) \\ &= Cov(X_t, X_{t+h}) + Cov(X_t, W_{t+h}) + Cov(W_t, X_{t+h}) + Cov(W_t, W_{t+h}) \\ &= Cov(X_t, X_{t+h}) + Cov(W_t, W_{t+h}) \\ &= \begin{cases} \frac{\sigma_z^2}{1-\phi^2} + \sigma_w^2, & h = 0 \\ \frac{\sigma_z^2 \phi^h}{1-\phi^2}, & h > 0 \end{cases}, \end{aligned}$$

where we have used that the AR(1) process has $Cov(X_t, X_{t+h}) = \frac{\sigma_z^2 \phi^h}{1-\phi^2}, h \geq 0$.

b)

$Cov(U_t, U_{t+h}) = Cov(Y_t, Y_{t+h}) - \phi Cov(Y_t, Y_{t-1+h}) - \phi Cov(Y_{t-1}, Y_{t+h}) + \phi^2 Cov(Y_{t-1}, Y_{t-1+h})$

We first look at the case $h = 0$

$$\begin{aligned}\gamma(0) &= Cov(U_t, U_t) = Cov(Y_t, Y_t) - \phi Cov(Y_t, Y_{t-1}) - \phi Cov(Y_{t-1}, Y_t) + \phi^2 Cov(Y_{t-1}, Y_{t-1}) \\ &= \cdots = \sigma_z^2 + \sigma_w^2(1 + \phi^2)\end{aligned}$$

For $h = 1$

$$\begin{aligned}\gamma(1) &= Cov(U_t, U_{t+1}) = Cov(Y_t, Y_{t+1}) - \phi Cov(Y_t, Y_t) - \phi Cov(Y_{t-1}, Y_{t+1}) + \phi^2 Cov(Y_{t-1}, Y_t) \\ &= \cdots = -\phi \sigma_z^2\end{aligned}$$

For $h > 1$

$$\begin{aligned}\gamma(h) &= Cov(U_t, U_{t+h}) = Cov(Y_t, Y_{t+h}) - \phi Cov(Y_t, Y_{t-1+h}) - \phi Cov(Y_{t-1}, Y_{t+h}) \\ &+ \phi^2 Cov(Y_{t-1}, Y_{t-1+h}) = 0\end{aligned}$$

c) Since U_t is an MA(1) process, $U_t = V_t + \theta V_{t-1}$, where $\{V_t\} \sim WN(0, \sigma_v^2)$, and from before we have $U_t = Y_t - \phi Y_{t-1}$, so the ARMA equation becomes

$$Y_t - \phi Y_{t-1} = V_t + \theta V_{t-1},$$

where ϕ is the same as before.
To find the parameters θ and σ_v^2, we use $\gamma(0)$,

$$\begin{gathered}Cov(U_t, U_t) = Cov(V_t, V_t) + \theta^2 Cov(V_{t-1}, V_{t-1}) \\ \downarrow \\ \sigma_z^2 + \sigma_w^2(1 + \phi^2) = \sigma_v^2 + \theta^2 \sigma_v^2\end{gathered}$$

and $\gamma(1)$

$$\begin{gathered}Cov(U_t, U_{t+1}) = Cov(V_t + \theta V_{t-1}, V_{t+1} + \theta V_t) \\ \downarrow \\ -\phi \sigma_z^2 = \theta \sigma_v^2\end{gathered}$$

To find θ, we must solve

$$\frac{\theta}{1 + \theta^2} = \frac{-\phi \sigma_w^2}{\sigma_z^2 + \sigma_w^2(1 + \phi^2)},$$

and then σ_v^2 can be obtained from $\sigma_v^2 = -\frac{\phi \sigma_w^2}{\theta}$.

Problem 2.13

a) Assume an AR(1)-model

$$X_t = \phi X_{t-1} + Z_t.$$

Since $\rho(h) = \phi^h, (h > 0)$ for an AR(1)-model, and it has been observed $\hat{\rho(2)} = 0.145$, we assume that $\phi^2 << 1$. Using Bartlett's formula,

$$\text{Var}[\hat{\rho(1)}] \approx \frac{1}{n}(1 - \phi^2)$$

and

$$\text{Var}[\hat{\rho(2)}] \approx \frac{1}{n}(1 - \phi^2)(1 + 3\phi^2)$$

That is, 95% confidence bounds for $\rho(1)$ are approximately

$$\hat{\rho(1)} \pm \frac{1.96}{\sqrt{n}}\sqrt{1 - \phi^2}$$

Correspondingly, 95% confidence bounds for $\rho(2)$ are approximately

$$\hat{\rho(2)} \pm \frac{1.96}{\sqrt{n}}\sqrt{(1 - \phi^2)(1 + 3\phi^2)}$$

With $\phi = \hat{\phi} = \hat{\rho(1)}$, $n = 100$, $\hat{\rho(1)} = 0.438$, $\hat{\rho(2)} = 0.145$, these bounds become 0.262, 0.614 for $\rho(1)$ and -0.073, 0.369 for $\rho(2)$. These values are not consistent with $\phi = 0.8$, since both $\rho(1) = 0.8$ and $\rho(2) = 0.64$ are outside these bounds.

b)Assume an MA(1)-model

$$X_t = Z_t + \theta Z_{t-1}.$$

Using Bartlett's formula,

$$\text{Var}[\hat{\rho(1)}] \approx \frac{1}{n}(1 - 3\rho(1)^2 + 4\rho(1)^4)$$

and

$$\text{Var}[\hat{\rho(2)}] \approx \frac{1}{n}(1 + 2\rho(1)^2)$$

That is, 95% confidence bounds for $\rho(1)$ are approximately

$$\hat{\rho(1)} \pm \frac{1.96}{\sqrt{n}}\sqrt{1-3\rho(1)^2+4\rho(1)^4}$$

Correspondingly, 95% confidence bounds for $\rho(2)$ are approximately

$$\hat{\rho(2)} \pm \frac{1.96}{\sqrt{n}}\sqrt{1+2\rho(1)^2}$$

With the numbers as in a), these bounds become 0.290, 0.586 for $\rho(1)$ and -0.082, 0.378 for $\rho(2)$. $\theta = 0.6$ leads to $\rho(1) = \frac{\theta}{1+\theta 2} = 0.4412, \rho(2) = 0$. If follows that the confidence bounds are consistent with these two values, and the data are therefore consistent with the MA(1)- model with $\theta = 0.6$

Problem 2.15

Let $\hat{X}_{n+1} = P_n X_{n+1} = a_0 + a_1 X_n + \cdots + a_n X_1$. We may assume that $\mu_X(t) = 0$. Let $S(a_0, ..., a_n) = \mathrm{E}[(X_{n+1} - \hat{X}_{n+1})^2]$ and minimize this with respect to $a_0, ..., a_n$.

$$\begin{aligned} S(a_0, ..., a_n) &= \mathrm{E}[(X_{n+1} - \hat{X}_{n+1})^2] \\ &= \mathrm{E}[(X_{n+1} - a_0 - a_1 X_n - \cdots - a_n X_1)^2] \\ &= a_0^2 - 2a_0 \mathrm{E}[X_{n+1} - a_1 X_n - \cdots - a_n X_1] \\ &+ \mathrm{E}[(X_{n+1} - a_1 X_n - \cdots - a_n X_1)^2] \\ &= a_0^2 + \mathrm{E}[(X_{n+1} - a_1 X_n - \cdots - a_n X_1)^2] \end{aligned}$$

where $\mathrm{E}[X_{n+1} - a_1 X_n - \cdots - a_n X_1] = 0$ from the properties of $P_n X_{n+h}$.

Differentiation with respect to a_i gives

$$\begin{aligned} \frac{\partial S}{\partial a_0} &= 2a_0 \\ \frac{\partial S}{\partial a_i} &= -2\mathrm{E}[(X_{n+1} - a_1 X_n - \cdots - a_n X_1) X_{n+1-i}], \quad i = 1, ..., n \end{aligned}$$

Putting the partial derivatives equal to zero, we get that $S(a_0, ..., a_n)$ is minimized if

$$\begin{aligned} &a_0 = 0 \\ &\mathrm{E}[(X_{n+1} - \hat{X}_{n+1}) X_k] = 0, \quad k = 1, ..., n. \end{aligned}$$

Plugging in the expression for X_{n+1} we get that for $k = 1, ..., n$,

$$\mathrm{E}[(\phi_1 X_n + \cdots + \phi_p X_{n-p+1} + Z_{n+1} - a_1 X_n - \cdots - a_n X_1) X_k] = 0$$

This is clearly satisfied if we let

$$\begin{cases} a_i = \phi_i, & 1 \le i \le p \\ a_i = 0, & i > p \end{cases}$$

Since the best linear predictor is unique, this is the one. The mean square error is

$$\mathrm{E}[(X_{n+1} - \hat{X}_{n+1})2] = \mathrm{E}[Z_{n+1}^2] = \sigma^2.$$

Problem 2.18

Given the MA(1) process $X_t = Z_t - \theta Z_{t-1}$, where $|\theta| < 1$, and $Z_t \sim WN(0, \sigma^2)$. Represented as an AR(∞) process, it assumes the form

$$Z_t = X_t + \theta X_{t-1} + \theta^2 X_{t-2} + ...$$

Setting $t = n+1$ in the last equation and applying $\hat{P}_n$ to each side, leads to the result

$$\hat{P}_n X_{n+1} = -\sum_{j=1}^{\infty} \theta^j X_{n+1-j} = \theta Z_n$$

Prediction error = $X_{n+1} - \hat{P}_n X_{n+1} = Z_{n+1}$. Hence, MSE = $\mathrm{E}[Z_{n+1}^2] = \sigma^2$.

Problem 2.19

The given MA(1)-model is $X_t = Z_t - Z_{t-1} : t \in Z$, where $Z_t \sim WN(0, \sigma^2)$. The vector $\mathbf{a} = (a_1, ..., a_n)^T$ of the coefficients that provide the best linear predictor (BLP) of X_{n+1} in terms of $X = (X_n, ..., X_1)^T$ satisfies the equation

$$\Gamma_n \mathbf{a} = \gamma_n$$

where the covariance matrix $\Gamma_n = Cov(\mathbf{X}, \mathbf{X})$ and $\gamma_n = Cov(X_{n+1}, \mathbf{X}) = (\gamma(1), ..., \gamma(n))^T$. Since $\gamma(0) = 2\sigma^2$, $\gamma(1) = -\sigma^2$, $\gamma(h) = 0$ for $|h| > 1$, it follows that

$$\Gamma_n = \sigma^2 \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1- & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{bmatrix}$$

and $\gamma_n = \sigma^2(-1, 0, ..., 0)^T$.

By solving the system $\Gamma_n \mathbf{a} = \gamma_n$, (by for example looking at a finite n or by performing Gauss elimination), the solution is given as follows

$$a_j = \frac{i}{n+1} - 1$$

Hence it is obtained that

$$P_n X_{n+1} = \sum_{j=1}^{n} (\frac{i}{n+1} - 1) X_{n+1-j}$$

The mean square error is

$$\mathrm{E}[(X_{n+1} - P_n X_{n+1})^2] = \gamma(0) - \mathbf{a}^T \gamma_n = 2\sigma^2 + a_1 \sigma^2 = \sigma^2 \left(1 + \frac{1}{n+1}\right)$$

Problem 2.20

We have to prove that $\mathrm{Cov}(X_n - \hat{X}_n, X_j) = \mathrm{E}[(X_n - \hat{X}_n)X_j] = 0$ for $j = 1, ..., n-1$. This follows from equations (2.5.5) for suitable values of n and h with $a_0 = 0$ (since we may assume that $\mathrm{E}[X_n] = 0$). This clearly implies that

$$\mathrm{E}[(X_n - \hat{X}_n)(X_k - \hat{X}_k)] = 0$$

for $k = 1, ..., n-1$, since $\hat{X}_k$ is a linear combination of $X_1, ..., X_{k-1}$.

$$
\begin{aligned}
\gamma(0) &= 1 + 0.3^2 + 0.4^2 = 1.25 \\
\gamma(1) &= 0.3 - 0.4 \cdot 0.3 = 0.18 \\
\gamma(2) &= -0.4 \\
\gamma(h) &= 0, \quad h > 2 \\
\gamma(-h) &= \gamma(h)
\end{aligned}
$$

b)

$$Y_t = \tilde{Z}_t - 1.2\tilde{Z}_{t-1} - 1.6\tilde{Z}_{t-2}$$

$$
\begin{aligned}
\gamma(0) &= 0.25(1 + 1.2^2 + 1.6^2) = 1.25 \\
\gamma(1) &= 0.25(-1.2 + 1.6 \cdot 1.2) = 0.18 \\
\gamma(2) &= -1.6 \cdot 0.25 = -0.4 \\
\gamma(h) &= 0, \quad h > 2 \\
\gamma(-h) &= \gamma(h)
\end{aligned}
$$

That is, we obtain the same ACVF as in a).

Exercise 2.5

$\sum_{j=1}^{\infty} \theta^j X_{n-j}$ converges absolutely (with probability 1) since

$$
\begin{aligned}
E[\sum_{j=1}^{\infty} |\theta|^j |X_{n-j}|] &\leq \sum_{j=1}^{\infty} |\theta|^j E[|X_{n-j}|] \\
&\leq \sum_{j=1}^{\infty} |\theta|^j \sqrt{\gamma(0) + \mu^2} \quad \text{by Cauchy-Schwartz inequality} \\
&< \infty \quad \text{since} |\theta| < 1
\end{aligned}
$$

That is, $\sum_{j=1}^{\infty} |\theta|^j |X_{n-j}| < \infty$ with probability 1.

Mean square convergence of $S_m = \sum_{j=1}^{m} \theta^j X_{n-j}$ as $m \to \infty$ can be verified by invoking Cauchy's criterion. For $m > k$

$$\begin{aligned} E[|S_m - S_k|^2] &= E[(\sum_{j=k+1}^{m} \theta^j X_{n-j})^2] \\ &= \sum_{i=k+1}^{m} \sum_{j=k+1}^{m} \theta^{i+j} E[X_{n-i} X_{n-j}] \end{aligned}$$

$$\begin{aligned} E[|S_m - S_k|^2] = E[(\sum_{j=k+1}^{m} \theta^j X_{n-j})^2] = \sum_{i=k+1}^{m} \sum_{j=k+1}^{m} \theta^{i+j} E[X_{n-i} X_{n-j}] \\ = \sum_{i=k+1}^{m} \sum_{j=k+1}^{m} \theta^{i+j} \big(\gamma(i-j) + \mu^2\big) \\ \leq \sum_{i=k+1}^{m} \sum_{j=k+1}^{m} |\theta|^{i+j} \big(\gamma(0) + \mu^2\big) = \big(\gamma(0) + \mu^2\big) \Big(\sum_{j=k+1}^{m} |\theta|^j \Big)^2 \\ \to 0 \quad \text{as} \quad k, m \to \infty \end{aligned}$$

since $\sum_{j=1}^{\infty} |\theta|^j < \infty$. Hence, by Cauchy's mutual convergence criterion, mean square convergence is guaranteed.

Exercise 2.7

$$\begin{aligned} \frac{1}{1 - \phi z} &= \frac{-\frac{1}{\phi z}}{1 - \frac{1}{\phi z}} \\ &= -\frac{1}{\phi z} \Big(1 + \frac{1}{\phi z} + \frac{1}{(\phi z)^2} + \dots \Big) \\ &= -\sum_{j=1}^{\infty} (\phi z)^{-j} \end{aligned}$$

since $|\phi z| > 1$.

Exercise 2.8

$$X_t = \phi X_{t-1} + Z_t$$

$$\begin{aligned}X_t &= \phi X_{t-1} + Z_t\\ &= Z_t + \phi(Z_{t-1} + \phi X_{t-2})\\ &= \dots\\ &= Z_t + \phi Z_{t-1} + \dots + \phi^n Z_{t-n} + \phi^{n+1} X_{t-n-1})\end{aligned}$$

That is

$$X_t - \phi^{n+1} X_{t-n-1} = Z_t + \phi Z_{t-1} + \dots + \phi^n Z_{t-n}$$

First we calculate

$$\begin{aligned}\text{Var}(X_t - \phi^{n+1} X_{t-n-1}) &= \gamma(0)(1 + \phi^{2n+2}) - 2\phi^{n+1}\gamma(n+1)\\ &\leq \gamma(0)(1 + |\phi|^{2n+2} + 2|\phi|^{n+1}) = 4\gamma(0)\end{aligned}$$

if X_t is stationary and $|\phi| = 1$

Next we calculate

$$\text{Var}(Z_t + \phi Z_{t-1} + \dots + \phi^n Z_{t-n}) = n\sigma^2$$

if $|\phi| = 1$

Since clearly $n\sigma^2 > 4\gamma(0)$ for sufficiently large n, we have reached a contradiction. Hence X_t cannot be stationary if $|\phi| = 1$.

Exercise 2.10

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$$

where $\phi = \theta = 0.5$

According to Section 2.3, equation (2.3.3), we obtain that

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

where $\psi_0 = 1$, $\psi_j = (\phi + \theta)\phi^{j-1} = 0.5^{j-1}$ for $j = 1, 2, \dots$.

From Section 2.3, equation (2.3.5), we get

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$$

where $\pi_0 = 1$, $\pi_j = -(\phi + \theta)(-\theta)^{j-1} = -(-0.5)^{j-1}$ for $j = 1, 2, \dots$.

Agrees with the results from ITSM.

GT Exercises

Exercise 2

a Let's remember that the expected value can be see as an inner product. That is,

$$< X_n, Y_m >= E(X_n Y_m)$$

So, using inner product notation, we can make use of the argument in the proof preposition 2.1.2 in *Time Series Theory and Methods*:

$$\begin{aligned}
|< X_n, Y_m > - < X, Y >| &= |< X_n, Y_m > - < X, Y > + < X_n, Y > - < X_n, Y >| \\
&= |< X_n, Y_m - Y > + < X_n - X, Y >| \\
&\leq ||X_n||\,||Y_m - Y|| + ||X_n - X||\,||Y|| \quad \text{By Cauchy-Schwarz}
\end{aligned}$$

Now, given that $X_n \to X$ and $Y_m \to Y$, then we conclude $|< X_n, Y_m > - < X, Y >| \to 0$ as $m, n \to \infty$

b From exercise 1 we know that $E(X_n Y_m|W) = P_{M(W)} X_n Y_m$ and $\hat{E}(X_n Y_m|W) = P_{\bar{sp}(1,W)} X_n Y_m$. Now, based on the property (iv) of projections in *Time Series Theory and Methods*:

Let P_M denote the projection mapping onto a closed subspace M

$$P_{M(W)} X_n Y_m \to P_{M(W)} XY \quad if \quad ||X_n Y_m - XY|| \to 0$$

From part a we know that $||X_n Y_m - XY|| \to 0$, so

$$P_{M(W)} X_n Y_m \to P_{M(W)} XY \equiv E(X_n Y_m|W) \to E(XY|W)$$

$$P_{\bar{sp}(1,W)} X_n Y_m \to P_{\bar{sp}(1,W)} XY \equiv \hat{E}(X_n Y_m|W) \to \hat{E}(XY|W)$$

c – If $\sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$ exists $\implies \sum_{j=-\infty}^{\infty} |\psi|^2 < \infty$

If $\sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} = \lim_{n\to\infty} \sum_{j=-n}^{n} \psi_j Z_{t-j}$ exists, then $(\lim_{n\to\infty} \sum_{j=-n}^{n} \psi_j Z_{t-j})^2 = \lim_{n\to\infty} (\sum_{j=-n}^{n} \psi_j Z_{t-j})^2$ exists.

Taking the expected value of $\lim_{n\to\infty}(\sum_{j=-n}^{n}\psi_j Z_{t-j})^2 < \infty$, we get:

$$\begin{aligned}
\lim_{n\to\infty} E\big(\sum_{j=-n}^{n}\psi_j Z_{t-j}\big)^2 &= \lim_{n\to\infty} E\big(\sum_{j=-n}^{n}\sum_{k=-n}^{n}\psi_j\psi_k Z_{t-j}Z_{t-k}\big)\\
&= \lim_{n\to\infty}\sum_{j=-n}^{n}\sum_{k=-n}^{n}\psi_j\psi_k E(Z_{t-j}Z_{t-k})\\
&= \lim_{n\to\infty}\sum_{j=-n}^{n}|\psi_j|^2\sigma^2 \quad \text{since } E(Z_{t-j}Z_{t-k}) = 0 \text{ for } t \neq k
\end{aligned}$$

Then,

$$\lim_{n\to\infty}\sum_{j=-n}^{n}|\psi_j|^2 < \infty$$

– $\sum_{j=-\infty}^{\infty}|\psi|^2 < \infty \implies \sum_{j=-\infty}^{\infty}\psi_j Z_{t-j}$ exists

$$\begin{aligned}
E\Bigg(\sum_{j=-\infty}^{\infty}\psi_j Z_{t-j}\Bigg)^2 &= \lim_{n\to\infty} E\Bigg(\sum_{j=-n}^{n}\psi_j Z_{t-j}\Bigg)^2\\
&= \lim_{n\to\infty}\sum_{j=-n}^{n}\psi_j^2\sigma^2\\
&= \sum_{j=-\infty}^{\infty}|\psi_j|^2\sigma^2 < \infty
\end{aligned}$$

Thus,$\sum_{j=-\infty}^{\infty}\psi_j Z_{t-j}$ exists.

d First of all, let's proof the convergence in mean square by making use of the Cauchy criterion. In order to do it, we will prove:

$$E(W_m - W_n)^2 \to 0 \quad \text{as } m, n \to \infty$$

Let's assume $m > n > 0$. Then,

$$
\begin{aligned}
E\left(\sum_{j=-\infty}^{\infty} \psi_j Y_{m-j} - \sum_{k=-\infty}^{\infty} \psi_k Y_{n-k}\right)^2 &= E\left(\sum_{j=n+1}^{m} \psi_j Y_j\right)^2 \\
&= E\left(\sum_{j=n+1}^{m} \sum_{k=n+1}^{m} \psi_j \psi_k Y_j Y_k\right) \\
&= \sum_{j=n+1}^{m} \sum_{k=n+1}^{m} \psi_j \psi_k \gamma(k-j) \\
&\leq \sum_{j=n+1}^{m} \sum_{k=n+1}^{m} |\psi_j|\,|\psi_k|\,|\gamma(k-j)| \\
&\leq \left(\sum_{j=n+1}^{m} |\psi_j|\right)^2 \gamma(0)
\end{aligned}
$$

which converges to 0 as $m, n \to \infty$ since $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$

Now, we can prove that W converges absolutely with probability one.

$$
\begin{aligned}
E|W| &= E\left|\sum_{j=-\infty}^{\infty} \psi_j Y_{t-j}\right| \\
&\leq \sum_{j=-\infty}^{\infty} |\psi_j|\, E|Y_{t-j}| \\
&= \sum_{j=-\infty}^{\infty} |\psi_j|\, c < \infty
\end{aligned}
$$

Given the stationarity of Y_t, we can state

$$
E|Y_t| \leq (E|Y_t|^2)^{1/2} = c
$$

e. *Linearity:*

We aim to prove: $P_M(\alpha X + \beta Y) = \alpha P_M(X) + \beta P_M(Y)$.

Since M is a linear subspace of $\mathcal{H}$, we know $\alpha P_M(X) + \beta P_M(Y) \in M$

As well,

$$\alpha X + \beta Y - (\alpha P_M(X) + \beta P_M(Y)) = \alpha(X - P_M(X)) + \beta(Y - P_M(Y))$$

By properties of projections, $(X - P_M(X)) \in M^\perp$ and $(Y - P_M(Y)) \in M^\perp$. Thus, $\alpha(X - P_M(X)) + \beta(Y - P_M(Y)) \in M^\perp$ because $M^\perp$ is a linear subspace of $\mathcal{H}$.

So, we can represent $\alpha X + \beta Y$ as the sum of an element of M and an element of $M^\perp$:

$$\alpha X + \beta Y = \alpha(X - P_M(X)) + \beta(Y - P_M(Y)) + -(\alpha P_M(X) + \beta P_M(Y))$$

And given that, the representation

$$X = P_M(X) + (I - P_M)X \quad P_M(X) \in M \quad (I - P_M(X)) \in M^\perp$$

is unique for each $X \in \mathcal{H}$, we can conclude $\alpha(X - P_M(X)) + \beta(Y - P_M(Y))$ is $P_M(\alpha X + \beta Y)$.

<u>*Continuity:*</u>

Now we aim to prove that if $||X_n - X|| \to 0$ then $P_M(X_n) \to P_M(X)$

First of all, let's see that $||X||^2 = ||P_M(X)||^2 + ||(I - P_M)X||^2$. By properties of projections $X = P_M(X) + (I - P_M)X$. Thus,

$$\begin{aligned} ||X||^2 =< X, X > &=< P_M X + (I - P_M)X, P_M X + (I - P_M)X > \\ &=< P_M X, P_M X > + < P_M X, (I - P_M)X > + < (I - P_M)X, P_M X > \\ &+ < (I - P_M)X, (I - P_M)X > \\ &=< P_M X, P_M X > + < (I - P_M)X, (I - P_M)X > \\ &= ||P_M(X)||^2 + ||(I - P_M)X||^2 \end{aligned}$$

Since $P_M X$ and $(I - P_M)X$ are orthogonal.

Thus,

$$||X_n - X||^2 = ||P_M(X_n - X)||^2 + ||(I - P_M)(X_n - X)||^2$$

which let us conclude $||P_M(X_n - X)||^2 \leq ||X_n - X||^2$.

Thus, if $||X_n - X||^2 \to 0$ then $||P_M(X_n - X)|| = ||P_M(X_n) - P_M(X)|| \to 0$