# TMA4285 Time series models
# Solution to exercise 9, autumn 2020

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## Problem 7.1

From the process we have $Z_t = \sqrt{h_t}e_t$ so that $Z_t^4 = h_t^2 e_t^4$.

$$E(Z_t^4) = E(e_t^4 h_t^2) = E(e_t^4)E(h_t^2) = (Var(e_t^2) + E(e_t^2)^2)E(h_t^2) = 3E(h_t^2)$$

since $Var(e_t^2) = 2$ and $E(e_t^2) = 1$. We use that $h_t = Z_t^2/e_t^2 = (\alpha_0 e_t^2 + \alpha_1 e_t^2 Z_{t-1}^2)/e_t^2 = \alpha_0 + \alpha_1 Z_{t-1}^2$. Then

$$\begin{aligned}
E(Z_t^4) &= 3E(\alpha_0^2 + 2\alpha_0\alpha_1 Z_{t-1}^2 + \alpha_1^2 Z_{t-1}^4) \\
E(Z_t^4) &= 3\alpha_0^2 + 6\alpha_0\alpha_1\frac{\alpha_0}{1-\alpha_1} + 3\alpha_1^2 E(Z_{t-1}^4) \\
E(Z_t^4)(1-3\alpha_1^2) &= \frac{(1-\alpha_1)3\alpha_0^2 + 6\alpha_0^2\alpha_1}{1-\alpha_1} \\
E(Z_t^4) &= \frac{3\alpha_0^2(1+\alpha_1)}{(1-3\alpha_1^2)(1-\alpha_1)}
\end{aligned}$$

where we have used that $E(Z_t^2) = \frac{\alpha_0}{1-\alpha_1}$. This equation can only be satisfied if $0 < \alpha_1^2 < 1/3$, which becomes the condition for finite expectation.

## Problem 7.2

We insert $Z_t^2 = h_t e_t^2$ into $Y_t$

$$Y_t = \frac{h_t e_t^2}{\alpha_0} = e_t^2(1 + \frac{1}{\alpha_0}\sum_{i=1}^{p}\alpha_i Z_{t-i}^2) = e_t^2(1 + \frac{1}{\alpha_0}\sum_{i=1}^{p}\alpha_0\alpha_i Y_{t-i}) = e_t^2(1 + \sum_{i=1}^{p}\alpha_i Y_{t-i})$$

where we have used that $Y_{t-i}\alpha_0 = Z_{t-i}^2$.

Next, we are supposed to deduce that $\{Y_t\}$ has the same autocorrelation function as the AR(p) process $W_t$. In Problem 7.3 we will see that scaling does not change the correlation which means that $Y_t$ will have the same autocorrelation as $W_t$. We could also show this by multiplying $Y_t$ with $Y_{t-h}$ and take the expectation. Then we would get the same equation for autocovariance as for the AR($p$) process.

## Problem 7.3

a)

$$
\begin{aligned}
E(Z_t^2|Z_{t-1}^2, Z_{t-2}^2, ...) &= E(h_t e_t^2|Z_{t-1}^2, Z_{t-2}^2, ...) = E(\alpha_0 + \sum_{i=1}^{p} \alpha_i Z_{t-1}^2 + \sum_{j=1}^{p} \beta_j h_{t-1}) \\
&= \alpha_0 + \sum_{i=1}^{p} \alpha_i E(Z_{t-1}^2) + \sum_{j=1}^{p} \beta_j E(h_{t-1}) = \alpha_0 + \sum_{i=1}^{p} \alpha_i Z_{t-1}^2 + \sum_{j=1}^{p} \beta_j h_{t-1} \\
&= h_t
\end{aligned}
$$

where we have used that $E(e_t^2) = 1$.

b) By inserting $U_t = Z_t^2 - h_t$ into the expression for $Z_t^2$ and solving for $h_t$, we end up with the equation for $h_t$.

c) In this exercise there is an error in the book. It should be $V_t = \alpha_1 U_{t-1}$. Using this, $\alpha_j^* = \alpha_{j+1}/\alpha_1$ and $U_t = Z_t^2 - h_t$ in the expression for $h_t$ we end up with the initial equation for $h_t$.