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LØSNINGSFORSLAG
EXAM IN TMA4285 TIME SERIES MODELS

Monday 04 December 2006
Time: 09:00–13:00

Oppgave 1

Let X_t be the ARMA(1,1) process defined by

$$X_t - \theta X_{t-1} = Z_t + \theta Z_{t-1}, \tag{1}$$

where $|\theta| < 1$, $Z_t \sim \mathrm{WN}(0,1)$.

a) Give definitions of causality and invertibility. Is this process causal? Invertible? Why?

Solution. Let X_t be the ARMA(p,q) process defined by

$$\phi(B)X_t = \theta(B)Z_t, \quad Z_t \sim \mathrm{WN}(p,q),$$

where

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p$$

and

$$\theta(z) = 1 - \theta_1 z - \ldots - \theta_q z^q.$$

X_t is causal (by definition) if

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \quad \sum_{j=0}^{\infty} |\psi_j| < \infty$$

and invertible if

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}, \quad \sum_{j=0}^{\infty} |\pi_j| < \infty.$$

Causality is equivalent to the condition $\phi(z) \neq 0$ for $|z| \leq 1$. Invertibility is equivalent to the condition $\theta(z) \neq 0$ for $|z| \leq 1$. These conditions are evidently satisfied in the considered case, therefore X_t is causal and invertible.

b) Find coefficients ψ_j, $j = 0, 1, ...$ of the representation

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}.$$

(Hint: write out operator $(1-\theta B)^{-1}$ and apply it to both sides of (1))

Solution. We have

$$X_t = (1-\theta B)^{-1}(1+\theta B)Z_t.$$

It is easy to see that

$$(1-\theta B)^{-1} = \sum_{j=0}^{\infty} \theta^j B^j.$$

Indeed,

$$(1-\theta B)\sum_{j=0}^{\infty} \theta^j B^j = \sum_{j=0}^{\infty} \theta^j B^j - \sum_{j=0}^{\infty} \theta^{j+1} B^{j+1} = 1.$$

Now

$$\sum_{j=0}^{\infty} \theta^j B^j (1+\theta B) = \sum_{j=0}^{\infty} \theta^j B^j + \sum_{j=1}^{\infty} \theta^j B^j = 1 + \sum_{j=0}^{\infty} 2\theta^j B^j,$$

and therefore

$$\psi_0 = 1, \quad \psi_j = 2\theta^j, \ j = 1, 2, ...$$

c) Find the ACVF of X_t as follows. First obtain equations from which $\gamma(0)$ and $\gamma(1)$ can be found. Then for $k \geq 2$ express $\gamma(k)$ in terms of $\gamma(k-1)$.

Solution. Multiplying both sides of (1) by X_t, X_{t-1} and taking the expectation, we (taking into account that $EX_tZ_t = \psi_0$, $EX_tZ_{t-1} = \psi_1$, $EZ_tX_{t-1} = 0$, $\psi_0 = 1$, $\psi_1 = 2\theta$) obtain the following equations

$$\gamma(0) - \theta\gamma(1) = 1 + 2\theta^2,$$

$$-\theta\gamma(0) + \gamma(1) = \theta.$$

Solutions are

$$\gamma(0) = \frac{1+3\theta^2}{1-\theta^2}, \quad \gamma(1) = 2\theta\left(\frac{1+\theta^2}{1-\theta^2}\right).$$

Let $k \geq 2$. Multiplying both sides of (1) by X_{t-k} and taking the expectation, we obtain

$$\gamma(k) - \theta\gamma(k-1) = 0$$

(because $EZ_tX_{t-k} = EZ_{t-1}X_{t-k}$=0) i.e.

$$\gamma(k) = \theta\gamma(k-1) = 2\theta^k\left(\frac{1+\theta^2}{1-\theta^2}\right).$$

Oppgave 2

Let X_t be the AR(2) time series defined by

$$X_t - \phi(1+\phi)X_{t-1} + \phi^3 X_{t-2} = Z_t,$$

where $|\phi| < 1$, $Z_t \sim \mathrm{WN}(0, \sigma^2)$.

a) Prove that this process is causal.

Solution. $1 - \phi(1+\phi)z + \phi^3 z^2 = (1-\phi z)(1-\phi^2 z)$. Roots $z_1 = 1/\phi$ and $z_2 = 1/\phi^2$ satisfy $|z_{1,2}| > 1$.

b) Let $n > 1$. Find P_nX_{n+1} the best linear predictor of X_{n+1} in terms of $X_1, ..., X_n$.

Solution.

$$\begin{aligned} P_nX_{n+1} = P(X_{n+1}|X_1,...,X_n) &= P\big(\phi(1+\phi)X_n - \phi^3X_{n-1} + Z_{n+1}|X_1,...,X_n\big) = \\ &= P\big(\phi(1+\phi)X_n - \phi^3X_{n-1}|X_1,...,X_n\big) + P\big(Z_{n+1}|X_1,...,X_n\big) = \\ &= \phi(1+\phi)X_n - \phi^3X_{n-1} + EZ_{n+1} = \phi(1+\phi)X_n - \phi^3X_{n-1} \end{aligned}$$

Consider the process Y_t defined by

$$Y_t = \left(1 - \frac{1}{2}B^2\right)X_t.$$

c) Show that Y_t is stationary.

Solution. Evidently $EY_t = 0$. Denote the ACVF of X_t by $\gamma(h)$. Then

$$Cov(Y_{t+h}, Y_t) = EY_{t+h}Y_t = E\left[\left(X_{t+h} - \frac{1}{2}X_{t+h-2}\right)\left(X_t - \frac{1}{2}X_{t-2}\right)\right] =$$

$$= \frac{5}{4}\gamma(h) - \frac{1}{2}\gamma(h-2) - \frac{1}{2}\gamma(h+2).$$

EY_t and $Cov(Y_{t+h}, Y_t)$ do not depend on t.

d) Let $n > 1$. Find $P(Y_{n+1}|X_1, ..., X_n)$ the best linear predictor of Y_{n+1} in terms of $X_1, ..., X_n$.

Solution.

$$P(Y_{n+1}|X_1, ..., X_n) = P\left(X_{n+1} - \frac{1}{2}X_{n-1}|X_1, ..., X_n\right) =$$

$$= P(X_{n+1}|X_1, ..., X_n) - \frac{1}{2}P(X_{n-1}|X_1, ..., X_n) =$$

$$= \phi(1+\phi)X_n - \phi^3 X_{n-1} - \frac{1}{2}X_{n-1} = \phi(1+\phi)X_n - \left(\phi^3 + \frac{1}{2}\right)X_{n-1}.$$

Oppgave 3

Establish which of the following two functions is the autocovariance function of a stationary process and which is not:

$$\gamma_1(h) = \begin{cases} 1 & \text{if } h = 0, \\ 1/2 & \text{if } h = \pm 1, \\ 2/3 & \text{if } h = \pm 2, \\ 0 & \text{otherwise.} \end{cases} \qquad \gamma_2(h) = \begin{cases} 0.4 & \text{if } h = 0, \\ -0.1 & \text{if } h = \pm 5, \\ -0.1 & \text{if } h = \pm 7, \\ 0 & \text{otherwise.} \end{cases}$$

Solution. The function

$$f_1(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma_1(h) = \frac{1}{2\pi}\left(1 + \cos\lambda + \frac{4}{3}\cos(2\lambda)\right)$$

takes negative values (for example $f(\pi/2) = -1/(6\pi)$) therefore $\gamma_1(h)$ is not the ACVF of a stationary process (Corollary 4.1.1, p. 114 of the textbook). The function

$$f_2(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma_2(h) = \frac{1}{2\pi}\left(0.4 - 0.2\cos(5\lambda) - 0.2\cos(7\lambda)\right) \geq 0$$

for any λ, therefore $\gamma_2(h)$ is not the ACVF of a stationary process.