TMA 4285 2007

Problem 1

a) W.S. $\mu_t = \mu$

$$\gamma(t_1, t_2) = \gamma(|t_2 - t_1|)$$

invertible if a process can be represented by a finite or infinite AR-process.

$$Z_t - \phi_2 Z_{t-2} = a_t$$

Characteristic equation $x^2 - \phi_2 = 0 \Rightarrow x = \pm\sqrt{\phi_2}$

The roots are less than one in absolute value for $|\phi_2| < 1$

and the process is stationary for $|\phi_2| < 1$

$\{Z_t\}$ is always invertible.

b). We have
$$Z_{t+1} = \phi_2 Z_{t-1} + a_{t+1}$$
$$Z_{t+2} = \phi_2 Z_t + a_{t+2}$$
$$Z_{t+3} = \phi_2 Z_{t+1} + a_{t+3}$$
$$= \phi_2(\phi_2 Z_{t-1} + a_{t+1}) + a_{t+3}$$
$$= \phi_2^2 Z_{t-1} + \phi_2 a_{t+1} + a_{t+3}$$

$$Z_{t+4} = \phi_2 Z_{t+2} + a_{t+4}$$
$$= \phi_2(\phi_2 Z_t + a_{t+2}) + a_{t+4} = \phi_2^2 Z_t + \phi_2 a_{t+2} + a_{t+4}$$

We get.

$$Z_t(1) = \phi_2 Z_{t-1}, \qquad e_t(1) = a_{t+1}$$

$$Z_t(2) = \phi_2 Z_t, \qquad e_t(2) = a_{t+2}$$

$$Z_t(3) = \phi_2^2 Z_{t-1}, \qquad e_t(3) = \phi_2 a_{t+1} + a_{t+3}$$

$$Z_t(4) = \phi_2^2 Z_t, \qquad e_t(4) = \phi_2 a_{t+2} + a_{t+4}$$

Problem 2.

$$Z_t = \sum_{j=0}^{\infty} \psi_j a_{t-j} \qquad \sum_{j=0}^{\infty} |\psi_j| < \infty$$

$$\gamma(k) = E\left[\sum_{j=0}^{\infty} \psi_j a_{t-j} \sum_{s=0}^{\infty} \psi_s a_{t-k-s}\right]$$

$$= \sum_{j=0}^{\infty} \sum_{s=0}^{\infty} \psi_j \psi_s E(a_{t-j} a_{t-k-s})$$

$$\overset{s=j-k}{=} \sigma_a^2 \sum_{j=0}^{\infty} \psi_j \psi_{j-k}$$

b)

$$Z_t - \phi Z_{t-1} = a_t + \theta a_{t-1}, \qquad |\phi| < 1, \quad |\theta| < 1$$

$$Z_t = (1-\phi B)^{-1}(1+\theta B) a_t = \psi(B) a_t$$

$$(1+\theta B) = (1-\phi B)(\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots)$$

$$\psi_0 = 1$$

$$\psi_1 - \psi_0 \phi = \theta \implies \psi_1 = \phi + \theta.$$

$$\psi_2 - \phi \psi_1 = 0 \implies \psi_2 = \phi(\phi+\theta)$$

$$\psi_3 - \phi \psi_2 = 0 \implies \psi_3 = \phi^2(\phi+\theta)$$

$$\psi_k = \phi^{k-1}(\phi+\theta)$$

We get:

$$\gamma(0) = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 = \sigma^2\left(1 + (\phi+\theta)^2 + \phi^2(\phi+\theta)^2 + \phi^4(\phi+\theta)^2 + \cdots\right)$$

$$= \sigma^2\left(1 + \frac{(\phi+\theta)^2}{1-\phi^2}\right) = \frac{\sigma^2(1+2\phi\theta+\theta^2)}{1-\phi^2}$$

$$\gamma(1) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j-1} = \sigma^2\left(\psi_1 + \psi_2\psi_1 + \psi_3\psi_2 + \psi_4\psi_3 + \cdots\right)$$

$$= \sigma^2\left((\phi+\theta) + \phi(\phi+\theta)^2 + \phi^3(\phi+\theta)^2 + \phi^5(\phi+\theta)^2 + \cdots\right)$$

$$= \sigma^2(\phi+\theta)\left(1 + \phi(\phi+\theta) + \phi^3(\phi+\theta) + \phi^5(\phi+\theta) + \cdots\right)$$

$$= \sigma^2(\phi+\theta)\left(1 + \frac{\phi(\phi+\theta)}{1-\phi^2}\right) = \frac{\sigma^2(\phi+\theta)(1-\phi^2+\phi^2+\theta\phi)}{1-\phi^2}$$

$$= \frac{\sigma^2(\phi+\theta)(1+\theta\phi)}{1-\phi^2}$$

Problem 3

a)

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + a_t$$

$$\Rightarrow \quad Z_{t+1} = \phi_1 Z_t + \phi_2 Z_{t-1} + a_t$$

$$Z_{t+2|t+1} = \phi_1 Z_{t+1} + \phi_2 Z_t$$

$$= \phi_1 Z_{t+1|t} + \phi_2 Z_t + \phi_1 a_{t+1}$$

which gives

$$\begin{bmatrix} Z_{t+1|t+1} \\ Z_{t+2|t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \phi_2 & \phi_1 \end{bmatrix} \begin{bmatrix} Z_{t|t} \\ Z_{t+1|t} \end{bmatrix} + \begin{bmatrix} 1 \\ \phi_1 \end{bmatrix} a_{t+1}$$

$$Z_t = Y_t = [1, 0] \begin{bmatrix} Z_{t|t} \\ Z_{t+1|t} \end{bmatrix}$$

$$Z_t = \phi Z_{t-1} + a_t + \theta a_{t-1}$$

$$Z_{t+1} = \phi Z_t + \theta a_t + a_{t+1}$$

$$Z_{t+2/t+1} = \phi Z_{t+1} + \theta a_{t+1}$$

$$= \phi(\phi Z_t + \theta a_t) + \phi a_{t+1} + \theta a_{t+1}$$

$$= \phi(Z_{t+1/t}) + (\phi + \theta) a_{t+1}$$

$$\begin{bmatrix} Z_{t+1/t+1} \\ Z_{t+2/t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \phi \end{bmatrix} \begin{bmatrix} Z_{t+1t} \\ Z_{t+1/t} \end{bmatrix} + \begin{bmatrix} 1 \\ \phi+\theta \end{bmatrix} a_{t+1}$$

$$Y_t = [1, 0] \begin{bmatrix} Z_{t/t} \\ Z_{t+1/t} \end{bmatrix}$$

b) Generation of forecasts.
Estimation of signal when the time series is observed with noise.

c)

$$X_t = F X_{t-1} + V_t \qquad Cov(V_t) = Q$$

$$Y_t = G X_t + W_t \qquad Cov(W_t) = R$$

Define ~~$\mu_{t/t}$~~ $\mu_{s/t} = E\{X_s / Y_t, Y_{t-1}, \ldots)$

$$\Omega_{s/t} = Cov\{X_s / Y_t, Y_{t-1}, \ldots)$$

$$\mu_{t+k/t} = E[F X_{t+k-1} + V_{t+k} / Y_t, Y_{t-1}, \ldots)$$

$$= F \mu_{t+k-1/t}$$

$$\Omega_{t+k/t} = Cov[F X_{t+k-1} + V_{t+k} / Y_t, Y_{t-1}, \ldots)$$

$$= F \Omega_{t+k-1/t} F' + Q.$$

d)

$$F = \begin{bmatrix} 0 & 1 \\ \phi_2 & \phi_1 \end{bmatrix} =$$

$$|F - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ \phi_2 & \phi_1 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow -\phi_1 \lambda + \lambda^2 - \phi_2 = 0$$

or $\lambda^2 - \phi_1 \lambda - \phi_2 = 0$,

$|\lambda| < 1 \iff$ AR(2) process is weakly stationary