Norwegian University of Science and Technology
Department of Mathematical Sciences

English

Contact person during the exam:
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EXAM TMA4285 TIME SERIES MODELS

11. December 2008
Time: 09:00–13:00

Permitted aids: *Tabeller og formler i statistikk*, Tapir Forlag
K. Rottmann: *Matematisk formelsamling*
Calculator HP30S or CITIZEN SR-270X
One yellow, stamped A5 sheet with own formulas and notes.

The results of the exam available 10. January 2008

NB: All answers must be justified.

Notation used in this problem set:

- Z_t is white noise with variance σ^2, that is, $Z_t \sim \text{WN}(0, \sigma^2)$.
- B is *backshift*-operator, such that $B^j X_t \equiv X_{t-j}$, $j \in \mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$
- ACVF = autocovariance function, ACF = autocorrelation function.
- IID = independent, identically distributed.
- N(0,1) = normally distributed with mean value 0 and variance 1.0.

Problem 1

You will find the figures for this problem at the end of the problem set.

a) Assume that the time series X_t, $t \in \mathbf{Z}$, is an AR(1) process given by $X_t + \phi X_{t-1} = Z_t$, where Z_t denotes white noise. In Fig. 1 are shown the ACF for 4 different values of the parameter ϕ. Determine these four values in the following order: 1) Upper left hand figure, 2) Upper right hand figure, 3) Lower left hand figure, and 4) Lower right hand figure.

b) Assume that the time series Y_t, $t \in \mathbf{Z}$, is an MA(q) process. In Fig. 2 are shown plots of Y_t versus Y_{t-k} for $k = 1, 2, 3$. Look very carefully at these plots and try to decide what the value of q is.

c) Assume that an observed time series X_t representing the demand of electricity over a period of more than 30 years looks like the one shown in Fig. 3. If you wanted to try to fit an ARMA model to this time series, suggest the first steps you would take in your efforts to make such a fit. Explain why.

Problem 2

Assume that the time series X_t is an ARMA(2, 1) process defined by

$$\phi(B)\, X_t = \theta(B)\, Z_t\,; \quad t \in \mathbf{Z} \tag{1}$$

where the AR polynomial $\phi(z) = 1 - z + \phi^2 z^2$, and the MA polynomial $\theta(z) = 1 + \theta z$.

a) What are the requirements that the parameters ϕ and θ must satisfy for X_t to be a (stationary) ARMA(2, 1) process? Hint: It may help you to show that for $\phi \neq 0$, the AR polynomial has the roots

$$z_{1,2} = \frac{2}{1 \pm \sqrt{1 - 4\phi^2}}.$$

b) Which additional requirements must ϕ and θ satisfy for X_t to be a causal and invertible time series?

For the remaining part of this exam problem it is assumed that $0 < |\phi| < 1$ and $\theta = 0$. The following result is also cited: For an AR(2) process the ACVF can be expressed as $\gamma(h) = c_1 z_1^{-h} + c_2 z_2^{-h}$ for $h \geq 0$ when $z_1 \neq z_2$, where $\phi(z_j) = 0, \;, \; j = 1, 2$, while $\gamma(h) = (c_1 + c_2 h) z_1^{-h}$ for $h \geq 0$ when $z_1 = z_2$.

c) Depending on the value of ϕ, $\gamma(h)$ will display qualitatively different behaviour. Without determining the constants c_1 and c_2 explicitly in terms of ϕ and σ, write down the expressions for $\gamma(h)$ as it depends on ϕ. Make sure that $\gamma(h)$ is always a real function of h.

d) Determine the explicit expression for $\gamma(h)$ for $\phi^2 = 1/2$, and show that the ACF is given as follows,

$$\rho(h) = \frac{\cos\left(\frac{\pi}{4}|h| + b\right)}{2^{|h|/2}\cos(b)}, \quad h \in \mathbf{Z} \tag{2}$$

where $\tan b = 2/7$. (Hint: $\cos t = (e^{it} + e^{-it})/2$, $i = \sqrt{-1}$.)

Problem 3

An ARCH(1) process X_t is given as,

$$X_t = \sqrt{H_t}\,\varepsilon_t\,, \quad \varepsilon_t \sim \text{IID N}(0,1)\,, \tag{3}$$

where

$$H_t = \alpha_0 + \alpha_1 X_{t-1}^2\,, \quad (\alpha_0 > 0, \alpha_1 > 0). \tag{4}$$

It is assumed that $0 < \alpha_1 < 1$. It can then be shown that

$$H_t = \alpha_0\big(1 + \sum_{j=1}^{\infty} \alpha_1^j \varepsilon_{t-1}^2 \cdot \ldots \cdot \varepsilon_{t-j}^2\big). \tag{5}$$

a) Argue why ε_t and H_t are independent random variables for each t.

Determine the expressions for $\mathrm{E}[X_t]$, $\mathrm{E}[X_t^2]$.

It turns out that $\mathrm{E}[X_t^4]$ does not exist for every α_1 satisfying $0 < \alpha_1 < 1$. Find the expression for $\mathrm{E}[X_t^4]$ and give the values of α_1 for its existence. ($\mathrm{E}[\varepsilon_t^4] = 3$.)

b) Let the process η_t be defined as follows, assuming that $\mathrm{E}[X_t^4] < \infty$,

$$\eta_t = X_t^2 - H_t = (\varepsilon_t^2 - 1)H_t. \tag{6}$$

Show that $\eta_t \sim \text{WN}(0, \sigma_0^2)$, and determine σ_0^2.

c) Establish an ARMA(p,q) model for X_t^2, and identify p and q.

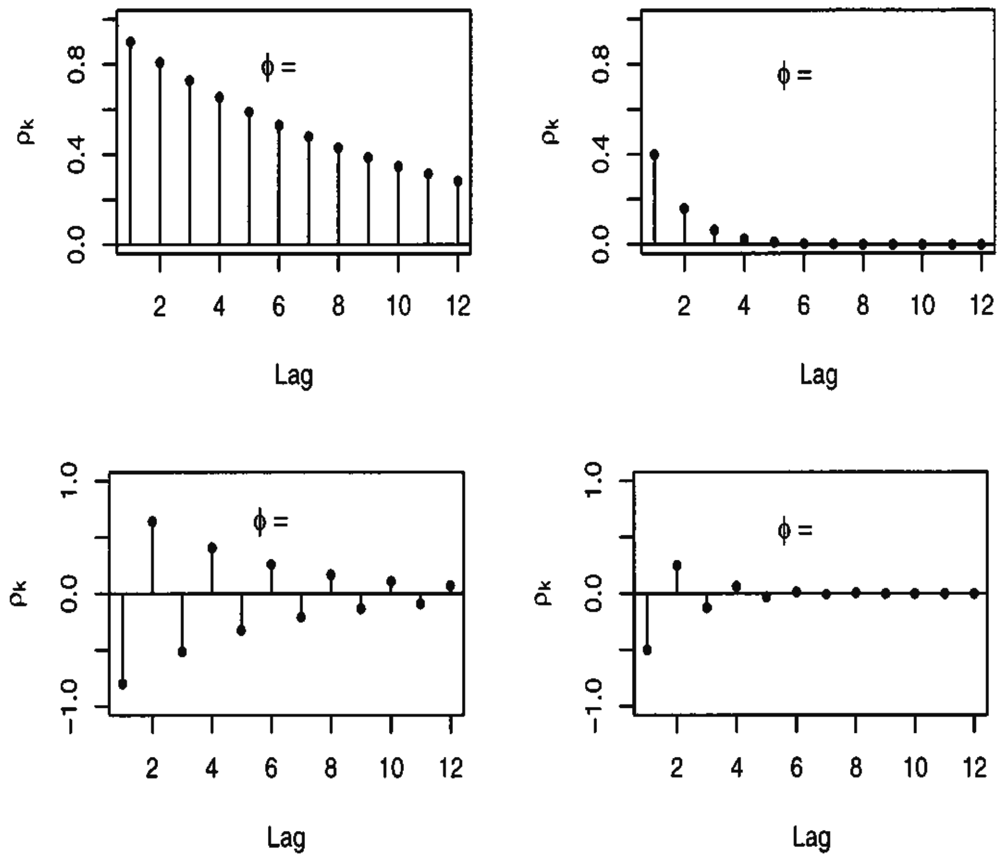

Figure 1: ACF for different values of ϕ

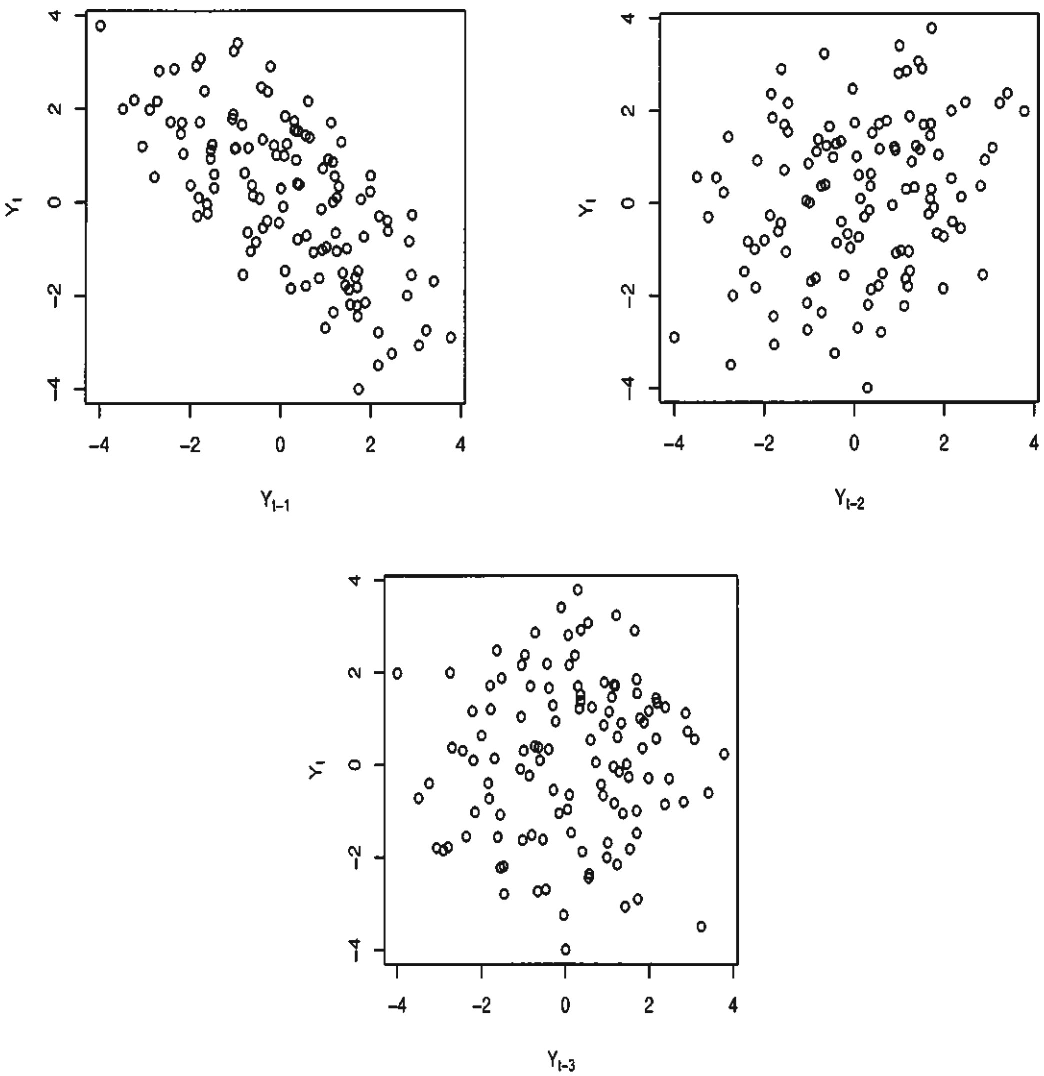

Figure 2:

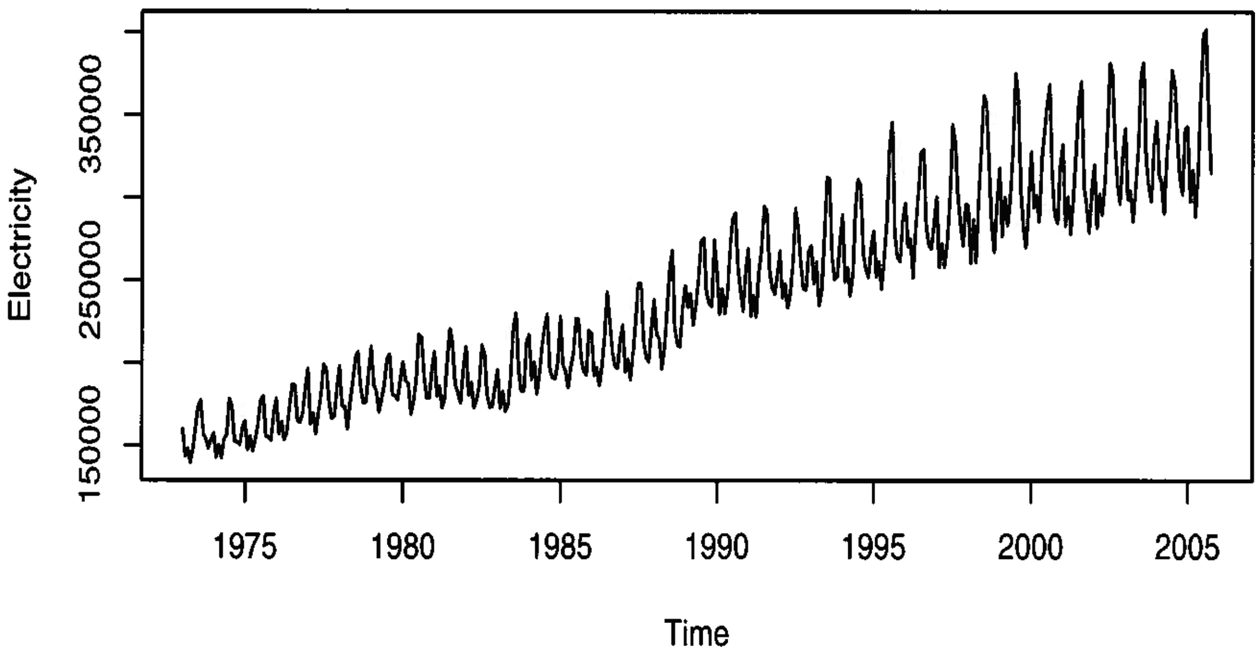

Figure 3: Demand for electricity.