TMA4285

Solutions Exam December 9. 2009

Problem 1

a)

$$E[Z_t] = \mu = \text{constant}.$$

$$E[(Z_t-\mu)^2] = \gamma(0) = \text{constant}.$$

$$E[(Z_t-\mu)(Z_s-\mu)] = \gamma(s,t) = \gamma(|s-t|)$$

Moving average processes are always a second order weakly stationary processes.

b)

$$1-\theta_1 B = 0 \quad \Rightarrow \quad B = \frac{1}{\theta_1}$$

$$1-\theta_{12} B^{12} = 0 \quad \Rightarrow \quad |B| = \left|\frac{1}{\theta_{12}}\right|^{\frac{1}{12}}$$

Both roots need to be greater than one in absolute value hence we must have: $|\theta_1| < 1$ and $|\theta_{12}| < 1$.

$$Z_t = a_t - \theta_1 a_{t-1} - \theta_{12} a_{t-12} + \theta_1\theta_{12} a_{t-13}$$

$$\gamma(0) = (1+\theta_1^2+\theta_{12}^2+\theta_1^2\theta_{12}^2)\sigma_a^2$$

$$\gamma(1) = E[(a_t-\theta_1 a_{t-1} - \theta_{12} a_{t-12} + \theta_1\theta_{12} a_{t-13})(a_{t-1} - \theta_1 a_{t-2} - \theta_{12} a_{t-13} + \theta_1\theta_{12} a_{t-14})]$$

$$= -\theta_1 \sigma_a^2 - \theta_1\theta_{12}^2 \sigma_a^2$$

$$\gamma(2) = \gamma(3) = \cdots = \gamma(10) = 0$$

$$\gamma(11) = E[(a_t - \theta_1 a_{t-1} - \theta_{12} a_{t-12} + \theta_1\theta_{12} a_{t-13})(a_{t-11} - \theta_1 a_{t-12} - \theta_{12} a_{t-23} + \theta_1\theta_{12} a_{t-24})]$$

$$= \theta_1\theta_{12}\sigma_a^2$$

$$\gamma(12) = E[(a_t - \theta_1 a_{t-1} - \theta_{12} a_{t-12} + \theta_1\theta_{12} a_{t-13})(a_{t-12} - \theta_1 a_{t-13} - \theta_{12} a_{t-24} + \theta_1\theta_{12} a_{t-25})]$$

$$= (-\theta_{12} - \theta_1^2\theta_{12})\sigma_a^2 = -\theta_{12}(1+\theta_1^2)\sigma_a^2.$$

$$\gamma(13) = E[(a_t - \theta_1 a_{t-1} - \theta_{12} a_{t-12} + \theta_1\theta_{12} a_{t-13})(a_{t-13} - \theta_1 a_{t-14} - \theta_{12} a_{t-25} + \theta_1\theta_{12} a_{t-26})] = \theta_1\theta_{12}\sigma_a^2$$

$$\gamma(k) = 0, \quad k \geq 14, \qquad \gamma(k) = \gamma(-k)$$

c)

$$\gamma(0) = (1 + 0.16 + 0.25 + 0.04)\sigma_a^2 = 1.45\sigma_a^2$$

$$\rho(1) = \frac{-0.5}{1.45} = -0.35$$

$$\rho(2) = \rho(3) = \dots = \rho(10) = 0$$

$$\rho(11) = \frac{0.2}{1.45} = 0.14$$

$$\rho(12) = \frac{-0.5(1+0.16)}{1.45} = -0.4$$

$$\rho(13) = \frac{0.2}{1.45} = 0.14$$

$$\rho(k) = 0, \qquad k \geq 14.$$

$$\rho(-k) = \rho(k)$$

This corresponds well with the plot. The greatest absolute values of the sample autocorrelation are for $k = 1$ and 12 and both are negative. For $k = 11$ and 13 the sample autocorrelations are positive though not significant.

d)

$$(1 - \pi_1 B - \pi_2 B^2 - \dots)(1 - \theta_1 B - \theta_{12} B^{12} + \theta_1\theta_{12} B^{13}) = 1$$

$$-\theta_1 - \pi_1 = 0 \quad \Rightarrow \quad \pi_1 = -\theta_1$$

$$-\pi_2 + \theta_1\pi_1 = 0 \quad \Rightarrow \quad \pi_2 = \theta_1\pi_1 = -\theta_1^2$$

$$\vdots$$

$$-\pi_{11} + \theta_1\pi_{10} = 0 \quad \Rightarrow \quad \pi_{11} = \theta_1\pi_{10} = -\theta_1^{11}$$

$$-\pi_{12} + \pi_{11}\theta_1 - \theta_{12} = 0 \quad \Rightarrow \quad \pi_{12} = -\theta_1^{12} - \theta_{12}.$$

$$-\pi_{13} + \pi_{12}\theta_1 + \underbrace{\pi_1\theta_{12} + \theta_1\theta_{12}}_{0} = 0 \quad \Rightarrow \quad \pi_{13} = \theta_1(-\theta_1^{12} - \theta_{12} + \theta_{12}) - \theta_1\theta_{12}$$

$$= -\theta_1^{13} - \theta_1\theta_{12} = \theta_1(-\theta_1^{12} - \theta_{12})$$

$$-\pi_{14} + \pi_{13}\theta_1 + \underbrace{\pi_2\theta_{12} - \pi_1\theta_1\theta_{12}}_{0} \Rightarrow \quad \pi_{14} = \theta_1^2(-\theta_1^{12} - \theta_{12})$$

$$\vdots$$

and $\pi_k = \theta_1^{k-12}(-\theta_1^{12} - \theta_{12}), \quad 12 \leq k \leq 23$

In general $\pi_k = \theta_1\pi_{k-1} + \theta_{12}(\pi_{k-12} - \pi_{k-13}\theta_1) \qquad k \geq 24.$

$-\pi_k =$

$$Z_{t+1|t} = \sum_{i=1}^{t} \pi_i Z_{t+1-i}$$

$$Z_{t+2|t} = \pi_1 Z_{t+1|t} + \sum_{i=2}^{t+1} \pi_i Z_{t+2-i}$$

$$Z_{t+l|t} = 0 \quad \text{for} \quad l \geq 14.$$

$$Z_{t+2|t} = \pi_1 \left(\pi_1 Z_t + \pi_2 Z_{t-1} + \pi_3 Z_{t-2} + \cdots \right)$$

$$\left. + \; \pi_2 Z_t + \pi_3 Z_{t-1} + \pi_4 Z_{t-2} + \cdots \right)$$

$$= \left(\pi_1^2 + \pi_2 \right) Z_t + \sum_{i=1}^{\infty} \left(\pi_1 \pi_{i+1} + \pi_{i+2} \right) Z_{t-i}$$

e) A model might be $(1-B)(1-B^{12})X_t = \theta_0 + (1-\theta_1 B)(1-\theta_{12}B^{12})a_t$

θ_0 is expected to be zero since we have both differencing and seasonal differencing.

$$X_t - X_{t-1} - X_{t-12} + X_{t-13} = \theta_0 + Z_t$$

$$\Rightarrow \quad X_{t+1/t} = X_t + X_{t-11} - X_{t-12} + \theta_0 + Z_{t+1/t}$$

$$X_{t+2/t} = X_{t+1/t} + X_{t-10} - X_{t-11} + \theta_0 + Z_{t+2/t}$$

$$(1-\theta_1 B)(1-\theta_{12}B^{12}) = (1-B)(1-B^{12})(1+\psi_1 B + \psi_2 B^2 + \dots)$$

$$\Rightarrow \quad -\theta_1 = \psi_1 - 1 \quad \Rightarrow \quad \psi_1 = 1-\theta_1$$

Prediction interval $\quad X_{t+2/t} \pm 1.96 \cdot \sigma_a \sqrt{1+(1-\theta_1)^2} = \underline{X_{t+2/t} \pm 1.96\sigma_a\sqrt{2-2\theta_1}}$

<u>Problem 3</u>

$Y_t - \phi_2 Y_{t-2} = a_t - \theta_1 a_{t-1} \,/\, Y_{t-k}$ and taking expectation gives

$$\gamma(0) - \phi_2\gamma(2) = \sigma_a^2 - \theta_1 E[Y_t a_{t-1}] = \sigma_a^2(1+\theta_1^2), \quad k=0$$

$$\gamma(1) - \phi_2\gamma(1) = -\theta_1\sigma_a^2, \quad k=1.$$

$$\gamma(k) - \phi_2\gamma(k-2) = 0, \quad k \geq 2.$$

Hence. $\gamma(0)(1-\phi_2^2) = \sigma_a^2(1+\theta_1^2) \quad \Rightarrow \quad \gamma(0) = \dfrac{\sigma_a^2(1+\theta_1^2)}{1-\phi_2^2}$

$$\gamma(1) = -\frac{\theta_1\sigma_a^2}{1-\phi_2}$$

$$\gamma(2) = \frac{\sigma_a^2(1+\theta_1^2)}{1-\phi_2^2}\cdot\phi_2.$$

$$\gamma(k) = \phi_2\gamma(k-2), \quad k \geq 2.$$

$$\gamma(k) = \gamma(-k)$$

$$\rho(1) = \frac{-\theta_1(1+\phi_2)}{1+\theta_1^2}$$

$$|\rho(1)| \le \frac{|\theta_1|}{1+\theta_1^2} \quad \text{for } \phi_2 \le 0.$$

Considering $g(\theta_1) = \frac{\theta_1}{1+\theta_1^2}$. $\quad g'(\theta_1) = \frac{1}{1+\theta_1^2} - \frac{2\theta_1^2}{(1+\theta_1^2)^2}$

$$= \frac{1+\theta_1^2 - 2\theta_1^2}{(1+\theta_1^2)^2} = \frac{1-\theta_1^2}{(1+\theta_1^2)^2} = 0 \quad \text{for } \theta_1 = \pm 1$$

and maximum and minimum - for $g(\theta_1) = \frac{1}{2}$ and $-\frac{1}{2}$ respectively.
for $|\theta_1| \le 1$

b)

$$Y_{t+1} = Y_{t+1|t} + a_t$$

$$Y_{t+2|t+1} = Y_t - \theta_1 a_{t+1}$$

System equation.

$$\Rightarrow \begin{bmatrix} Y_{t+1|t+1} \\ Y_{t+2|t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \phi_2 & 0 \end{bmatrix} \begin{bmatrix} Y_{t|t} \\ Y_{t+1|t} \end{bmatrix} + \begin{bmatrix} 1 \\ -\theta_1 \end{bmatrix} a_{t+1}$$

$$Y_t = [1, 0] \begin{bmatrix} Y_{t|t} \\ Y_{t+1|t} \end{bmatrix}$$

c) $x^2 - \phi_2 = 0 \Rightarrow x = \pm\sqrt{\phi_2}$.

$$f(l) = c_1(\sqrt{\phi_2})^l + c_2(-\sqrt{\phi_2})^l, \quad l \ge 1 - 2 + 1 = 0$$

1 step ahead conditional forecast error variance: $E_t(a_{t+1}^2) = \sigma_{t+1}^2 = \alpha_0 + \alpha_1 a_t^2$.

Two step ahead: $\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 \Rightarrow a_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \varepsilon_t \leftarrow$ white noise
such that $E_t(a_{t+1}^2) = \alpha_0 + \alpha_1 a_t^2$ and
$Var(e_t(2)) = E_t(a_{t+2}^2) + \psi_1^2(\alpha_0 + \alpha_1 a_t^2) = \alpha_0 + \alpha_1 E_t(a_{t+1}^2) + \theta_1^2(\alpha_0 + \alpha_1 a_t^2) = \alpha_0 + (\alpha_0 + \alpha_1 a_t^2)(\alpha_1 + \theta_1^2)$