Lf TMA 4285 2010

1 a)

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} Z_i = \bar{Z}$$

$$\hat{\rho}_k = \frac{\sum_{i=1}^{n-k}(Z_i-\bar{Z})(Z_{i+k}-\bar{Z})}{\sum_{i=1}^{n}(Z_i-\bar{Z})^2}$$

$$\hat{\gamma}(0) = \sum_{i=1}^{n}(Z_i-\bar{Z})^2/n$$

b)

MA - processes are always stationary

$$X^2 - 0.5X - 0.14 = 0 \Rightarrow X = 0.25 \pm \sqrt{(0.25)^2+0.14} \Rightarrow \begin{cases} X_1 = 0.7 \\ X_2 = -0.2 \end{cases}$$

Both roots are less than one i absolute value. Hence the process is invertible.

$$Z_t = \theta(B)\,a_t \Rightarrow a_t = \theta^{-1}(B)\,Z_t = \pi(B)\,Z_t$$

$$\Rightarrow \theta(B)\,\pi(B) = 1$$

Hence $(1-0.5B-0.14B^2)(1+\pi_1 B + \pi_2 B^2 + \pi_3 B^3 + \cdots) = 1$

Therefore $(\pi_1 - 0.5) = 0 \Rightarrow \pi_1 = 0.5$

$$\pi_2 - 0.5\pi_1 - 0.14 = 0 \Rightarrow \pi_2 = 0.5^2 + 0.14 = 0.39$$

$$\pi_3 - 0.5\pi_2 - 0.14\pi_1 = 0 \Rightarrow \pi_3 = 0.5\cdot 0.39 + 0.14\cdot 0.5 = 0.265$$

$$\pi_4 = 0.5\cdot 0.265 + 0.14\cdot 0.39 \approx 0.18.$$

$\pi_k = c_1(0.7)^k + c_2(-0.2)^k$ decreasing with k.

Likely to fit and AR(3) may be AR(4).

2a)

The ACF ~~shows~~ is decaying slowly

The PACF has a peak approximately equal to 1 at lag one.

The plot seems to have an increasing trend.

This suggests that the time series should be differenced.

With a start in t_0 we have:

$$X_{t_0+1} = X_{t_0} + \theta_0 + a_{t_0+1}$$

$$X_{t_0+2} = X_{t_0+1} + \theta_0 + a_{t_0+2} = X_{t_0} + \theta_0 + a_{t_0+1} + \theta_0 + a_{t_0+2} = X_{t_0} + 2\theta_0 + a_{t_0+1} + a$$

and $X_{t_0+k} = X_{t_0} + k\theta_0 + \sum_{i=1}^{k} a_{t_0+i}$ which shows that the process has a linear trend.

b)

$$Y_t = a_t - \theta_1 a_{t-1} + \theta_0.$$

$$\gamma_0 = E[(a_t - \theta_1 a_{t-1})(a_t - \theta_1 a_{t-1})] = (1+\theta_1^2)\sigma_a^2.$$

$$\gamma_1 = E[(a_t - \theta_1 a_{t-1})(a_{t-1} - \theta_1 a_{t-2})] = -\theta_1 \sigma_a^2 = \gamma_{-1}$$

$$\gamma_k = E[(a_t - \theta_1 a_{t-1})(a_{t-k} - \theta_1 a_{t-k-1})] = 0, \quad |k| \geq 2$$

$$Var(\bar{Y}) = \frac{1}{n^2}\left[\sum_{i=1}^{n} Var(Y_i) + 2\sum_{i=1}^{n-1} Cov(Y_i, Y_{i+1})\right] = \frac{\gamma_0}{n} + \frac{2(n-1)\gamma_1}{n^2}$$

$$= \frac{\sigma_a^2}{n}\left(1+\theta_1^2 - \frac{2(n-1)}{n}\theta_1\right) = \frac{\sigma_a^2}{n}\left(\frac{n(\theta_1-1)^2 + 2\theta_1}{n}\right) = \frac{\sigma_a^2}{n}\left((\theta_1-1)^2 + \frac{2\theta_1}{n}\right)$$

c)

A natural estimator is $\bar{y}$

With the given parameters $\hat{\sigma}_{\bar{y}} = \sqrt{\frac{0.001}{129}\left(0.4^2 + \frac{1.2}{129}\right)} = 0.00115$

Hence $\frac{\bar{y}}{\hat{\sigma}_{\bar{y}}} = \frac{\hat{\theta}_0}{\hat{\sigma}_{\bar{y}}} = \frac{0.0066}{0.00115} = 5.76$ and we conclude

that θ_0 is significant different from 0.

We conclude there has been a global warming.

d)

$\hat{X}_{131} = \hat{X}_{130}(1) = X_{130} - 0.6\,\hat{a}_{130} + \theta_0 = 0.57 - 0.6 \cdot 0.07 + 0.0066$

$= 0.57 - 0.042 + 0.0066 = 0.535$

Hence the predicted average global temperature in 2010 is 15.035

The forecast function for X_t is

$f(l) = C_0 + \theta_0 l \quad$ for $l \geq q+1-p-d = 2-1 = 1$

Hence. $f(1) = 0.535 = C_0 + 0.0066$ and $C_0 = 0.528$

Therefore the predicted global average temperature in 2100

is $14.5 + f(91) = 14.5 + 0.528 + 0.0066 \cdot 91 = 14.5 + 0.528 + 0.601$

$= \underline{15.629\ ^\circ C}$

We have $\psi(B)(1-B) = \theta(B)$

$$(1+\psi_1 B+\psi_2 B^2+\cdots)(1-B) = 1-\theta_1 B$$

$$\psi_1 - 1 = -\theta_1 \Rightarrow \psi_1 = 1-\theta_1$$

$$\psi_k - \psi_{k-1} = 0,\ k \geq 2 \Rightarrow \psi_k = \psi_{k-1} = \psi_1,\quad k\geq 2.$$

Hence $e_{2009}(91) = \sum_{i=0}^{90} \psi_i a_{2010+i}$

and $Var(e_{2009}(91)) = \sum_{i=0}^{90} \psi_i^2 \sigma_a^2 = \sigma_a^2(1+90(1-\theta_1)^2)$

$$= \sigma_a^2(1+90\cdot 0.4^2) = 15.4\,\sigma_a^2$$

95% prediction interval $15.63 \pm 1.96\sigma_a\sqrt{15.4} = (15.39,\ 15.87)$

e)

With $\phi_1 = -0.5$, $\phi_2 = -0.39$ and $\phi_3 = -0.265$

$(1-B)X_t = (1-0.5B-0.14B^2)a_t$ is an approximate ARIMA(3,1,0) model according to 1b)

Solving $(1+\psi_1 B+\psi_2 B^2+\cdots)(1-B) = 1-\theta_1 B-\theta_2 B^2$

we get.
$$\psi_1 - 1 = -\theta_1 \Rightarrow \psi_1 = 1-\theta_1 = 0.5$$

$$\psi_2 - \psi_1 = -\theta_2 \Rightarrow \psi_2 = \psi_1 - \theta_2 = 0.5-0.14 = 0.36.$$

$$\psi_k - \psi_{k-1} = 0 \quad k\geq 3 \Rightarrow \psi_k = \psi_2,\quad k\geq 3$$

hence. $Var(e_{2009}(91)) = \sigma_a^2(1+0.5^2+89(0.36)^2)$

$1+0.5^2+89\cdot 0.36^2 < 1+0.4^2+89\cdot 0.4^2$. The estimate for σ_a^2 is expected to smaller. Hence the uncertainty will be smaller.

f)
$$\begin{bmatrix} X_{t+1} \\ \beta_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} X_t \\ \beta_t \end{bmatrix} + \begin{bmatrix} a_{t+1} \\ e_{t+1} \end{bmatrix} \quad \text{Dynamic equation.}$$

$$Y_t = [1,0]\begin{bmatrix} X_t \\ \beta_t \end{bmatrix} \quad \text{Observation equation.}$$

$$\begin{bmatrix} X_{t+1|t} \\ \beta_{t+1|t} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_{t|t} \\ \beta_{t|t} \end{bmatrix} = \underline{A} \begin{bmatrix} X_{t|t} \\ \beta_{t|t} \end{bmatrix}$$

$$\begin{bmatrix} X_{2100|2009} \\ \beta_{2100|2009} \end{bmatrix} = A^{91} \begin{bmatrix} X_{2009|2009} \\ \beta_{2009|2009} \end{bmatrix} = \begin{bmatrix} 1 & 91 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_{2009} \\ \hat{\beta}_{2009} \end{bmatrix} = X_{2009} + 91 \cdot 0.0066$$

$$= \underline{15.67}$$