Solution TMA 4285, 2011

1a) $(1-\phi B)Z_t = (1-\theta B)a_t$ is second order w. s.

if $|\phi| < 1$

$Z_t = (1-\theta_1 B - \theta_2 B^2)a_t$ is always w. s since

MA-processes are always w. s.

For MA(2)

$Z_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} \mid Z_{t-k},\ k \geq 0$ and take

expectation gives.

$$\gamma_0 = E[(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})] = (1+\theta_1^2+\theta_2^2)\sigma_a^2$$

$$\gamma_1 = E[(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})(a_{t-1} - \theta_1 a_{t-2} - \theta_2 a_{t-3})] = \sigma_a^2(-\theta_1 + \theta_1\theta_2)$$

$$\gamma_2 = E[(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})(a_{t-2} - \theta_1 a_{t-3} - \theta_2 a_{t-4})] = -\theta_2\sigma_a^2$$

$\gamma_k = 0$ for $k \geq 3$

Hence $\rho_1 = -\dfrac{\theta_1 + \theta_1\theta_2}{1+\theta_1^2+\theta_2^2}$, $\rho_2 = -\dfrac{\theta_2}{1+\theta_1^2+\theta_2^2}$, $\rho_k = 0$ for $k \geq 3$

and $\rho_k = \rho_{-k}$.

b) $Z_t - \phi Z_{t-1} = a_t - \theta a_{t-1} \mid Z_{t-k}$ and take expectation gives

$k=0$: $\gamma_0 - \phi\gamma_1 = \sigma_a^2 - \theta E[a_{t-1}(\phi Z_{t-1} + a_t - \theta a_{t-1})] = \sigma_a^2 - \theta\sigma_a^2(\phi - \theta)$ (1)

$k=1$: $\gamma_1 - \phi\gamma_0 = -\theta\sigma_a^2$ (2)

$k \geq 2$ $\gamma_k - \phi\gamma_{k-1} = 0$

(2) $\Rightarrow$ $\gamma_1 = \phi \gamma_0 - \theta \sigma_a^2$ (2) which inserted in (1) gives

$$\gamma_0(1-\phi^2) = \sigma_a^2(1+\theta^2 - 2\phi\theta) \Rightarrow \gamma_0 = \frac{\sigma_a^2(1+\theta^2-2\phi\theta)}{1-\phi^2}$$

$$\gamma_1 = \sigma_a^2\left[\frac{\phi(1+\theta^2-2\phi\theta)}{1-\phi^2} - \theta\right] = \frac{\sigma_a^2}{1-\phi^2}(\phi + \phi\theta^2 - \theta - \theta\phi^2)$$

$$= \frac{\sigma_a^2}{1-\phi^2}(\phi-\theta)(1-\phi\theta)$$

Hence $\rho_1 = \dfrac{(\phi-\theta)(1-\phi\theta)}{1+\theta^2-2\phi\theta}$

Since $\rho_k = \phi \rho_{k-1}$ for $k \geq 2$ we get $\rho_k = \phi^{k-1}\rho_1$, $\rho_k = \rho_{-k}$

c)

ACF has two significant lags (the two first) and PACF has just the first. The PACF may point to an AR(1) but since $\frac{\hat{\rho}_1}{1} << \frac{\hat{\rho}_2}{\rho_1}$ an ARMA(1,1) may be more likely.

The two significant ACF lags can be explained by an MA(2) process (with negative θ_1 and θ_2 according to 1a).

d).

$$Z_t = (1-\phi B)^{-1}(1-\theta B)a_t = \psi(B)a_t$$

Hence $(1-\phi B)\psi(B) = 1-\theta B$

or $(1-\phi B)(1+\psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots) = 1-\theta B$

$\psi_1 - \phi = -\theta \Rightarrow \psi_1 = \phi - \theta$

$\psi_2 - \phi\psi_1 = 0 \Rightarrow \psi_2 = \phi(\phi-\theta)$

$\psi_k - \phi\psi_{k-1} = 0 \quad k \geq 2 \Rightarrow \psi_k = \phi^{k-1}(\phi-\theta)$

(3)

$\phi = 0.8$ and $\theta = 0.34$

$\Rightarrow \quad \psi_1 = 0.8 - 0.34 = 0.46 \qquad \psi_2 = 0.8 \cdot 0.46 = 0.37$

$\psi_3 = 0.8^2 \cdot 0.46 = 0.30 \qquad \psi_4 = 0.8^3 \cdot 0.46 = 0.24$

The ψ- weights are rather large for $k > 2$ and the MA(2) model is probably not a good approximation.

e)

Let $Z_t = \ln Y_t$

$\hat{Z}_t(1) = Z_t - \theta_1 a_t - \theta_2 a_{t-1} + \theta_0 = 2.1193 + 0.32(-0.00063) + 0.28(-0.00039) + 0.001$

≈ 2.1199

$\hat{Z}_t(2) = \hat{Z}_t(1) - \theta_2 a_t + \theta_0 = 2.1199 + 0.28(-0.00063) + 0.001 = 2.1207$

$Var_t(e(2)) = \sigma_a^2(1 + \psi_1^2) = \sigma_a^2(1 + (1-\theta_1)^2) = 2.74\sigma_a^2.$

Hence a 95% prediction interval is:

$$2.1207 \pm 1.96 \cdot 1.3 \cdot 10^{-3}\sqrt{2.74} = 2.1207 \pm 0.0042 = (2.1165,\ 2.1249)$$

f)

$\sigma_t^2 = a_t^2 - \varepsilon_t$ gives

$a_t^2 - \varepsilon_t = 0.65(a_{t-1}^2 - \varepsilon_{t-1}) + 0.17 a_{t-1}^2$

$\Rightarrow \quad a_t^2 - 0.82 a_{t-1}^2 = \varepsilon_t - 0.65\varepsilon_{t-1}$ — hvit støy

With $\phi = 0.82$ and $\theta = 0.65$ we get from b).

$$\rho_1 = \frac{0.17(1 - 0.82 \cdot 0.65)}{1 + 0.65^2 - 2 \cdot 0.82 \cdot 0.65} = 0.22, \qquad \rho_2 = 0.22 \cdot 0.82 = 0.18$$

i.e. a reasonable good fit.

$\pi(B)(1-0.65B) = (1-0.82B)$ gives.

$(1+\pi_1 B + \pi_2 B^2 + \pi_3 B^3 + \dots)(1-0.65B) = 1-0.82B$

Hence $\pi_1 - 0.65 = -0.82 \Rightarrow \pi_1 = -0.17$

$\pi_2 - 0.65\pi_1 = 0 \Rightarrow \pi_2 = -0.11$

The ARCH(2) approximation is $a_t^2 - 0.17 a_{t-1}^2 - 0.11 a_{t-2}^2 = \varepsilon_t$

For ARCH(2) models: $Var_t(e(2)) = E_t[a_{t+1}^2] + \psi_1^2 E_t[a_{t+2}^2]$

$E_t[a_{t+1}^2] \approx 0.17 a_t^2 + 0.11 a_{t-1}^2 = 0.17(-0.00063)^2 + 0.11(-0.00059)^2 < (1.3 \cdot 10^{-3})^2$

$E_t[a_{t+2}^2] \approx 0.17 E_t[a_{t+1}^2] + 0.11 a_t^2 = 0.17[0.17 a_t^2 + 0.11 a_{t-1}^2] + 0.11 a_t^2$

$\approx 0.14 a_t^2 + 0.019 a_{t-1}^2 < (1.3 \cdot 10^{-3})^2$

Hence the interval will be shorter.

g)

$\frac{y_t}{y_{t-1}} = 1 + \rho_t \Rightarrow \ln\left[\frac{y_t}{y_{t-1}}\right] \approx \rho_t$

or $\ln y_t - \ln y_{t-1} = \nabla \ln y_t \approx \rho_t$

$X_t = \nabla \ln y_t = (1-\theta_1 B - \theta_2 B^2) a_t + \theta_0 \approx \rho_t$ or

$X_t = (1-\theta_1 B - \theta_2 B^2) a_t + \theta_0$, is a reasonable model.

$E[X_t] = \theta_0$. Hence θ_0 is the expected return rate.

h) $P(L_L \le \ln(y_t) \le L_u) = 0.95 \Rightarrow P(e^{L_L} < y_t < e^{L_u}) = 0.95$

since $L(y_t)$ is strict monotone. Hence the 95% prediction interval

is (e^{L_L}, e^{L_u})

Let

$Z_t = \ln(Y_t)$. We have

$$Z_t - \phi_1 Z_{t-1} - \phi_2 Z_{t-2} - \phi_3 Z_{t-3} = (1-\theta_1 B - \theta_2 B^2) a_t + \theta_0$$

where $\phi_1 = 1$ and $\phi_2 = \phi_3 = 0$. Thereby.

$$\begin{bmatrix} Z_{t+1|t+1} \\ Z_{t+2|t+1} \\ Z_{t+3|t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Z_{t|t} \\ Z_{t+1|t} \\ Z_{t+2|t} \end{bmatrix} + \begin{bmatrix} 1 \\ \psi_1 \\ \psi_2 \end{bmatrix} a_{t+1}$$

$\psi_1 = 1 - \theta_1 \qquad \psi_2 = 1 - \theta_1 - \theta_2.$

$$Z_t = [1, 0, 0] \begin{bmatrix} Z_{t|t} \\ Z_{t+1|t} \\ Z_{t+2|t} \end{bmatrix}$$