Norwegian University of Science and Technology
Department of Mathematical Sciences

English

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EXAM IN TMA4285 TIME SERIES MODELS

Friday December 7th 2012
Time: 09:00 – 13:00

Permitted aids: Calculator HP30S, CITIZEN SR-270X or CITIZEN SR-270X College.
Statistiske tabeller og formler, Tapir forlag.
K. Rottman: Matematisk formelsamling.
One yellow, stamped A5 sheet with own handwritten formulas and notes.

The results of the exam are available Monday January 7th 2013.

Note: You should give reasons for all your answers!

Note: The Kalman filter equations are given in the last page of this problem set.

Problem 1

Consider an ARMA(1,2) model with zero mean,

$$(1-\varphi_1 B)z_t = (1-\theta_1 B - \theta_2 B^2)a_t,$$

where φ_1, θ_1 and θ_2 are parameters, B denotes the *backshift* operator, i.e. $B^k z_t = z_{t-k}$, and $\{a_t\}$ is a white noise process with zero mean and $\mathrm{Var}(a_t) = \sigma_a^2$.

a) What does it mean that $\{z_t\}$ is invertible?
Is $\{z_t\}$ invertible when $\varphi_1 = \frac{1}{2}$, $\theta_1 = \frac{1}{2}$ and $\theta_2 = -\frac{1}{4}$?

What does it mean that $\{z_t\}$ is covariance stationary?
Is $\{z_t\}$ covariance stationary when $\varphi_1 = \frac{1}{2}$, $\theta_1 = \frac{1}{2}$ and $\theta_2 = -\frac{1}{4}$?

In the following we assume $\{z_t\}$ to be both invertible and covariance stationary. As you know this imply in particular that $\{z_t\}$ has an MA(∞) representation, $z_t = \psi(B)a_t$.

b) Derive formulas for the coefficients $\{\psi_j\}_{j=0}^{\infty}$ and show that they can be expressed as

$$\psi_0 = 1, \psi_1 = \theta_1 - \varphi_1 \quad \text{and} \quad \psi_j = \varphi_1^{j-2}((\theta_1 - \varphi_1)\varphi_1 + \theta_2) \quad \text{for } j = 2, 3, \ldots.$$

c) From the MA(∞) representation of $\{z_t\}$ given above, show that the variance of z_t is given by

$$\gamma_0 = \sigma_a^2 \left[1 + (\theta_1 - \varphi_1)^2 + \frac{((\theta_1 - \varphi_1)\varphi_1 + \theta_2)^2}{1 - \varphi_1^2} \right].$$

From the same MA(∞) representation derive also a corresponding formula for γ_1.

Problem 2

A plot of an observed time series z_t is given in the upper left corner of Figure 1. In the same figure the estimated autocorrelation function and partial autocorrelation function for the observed time series are shown. The observed time series after differencing and corresponding estimated autocorrelation function and partial autocorrelation function for the differenced time series are given in the lower row of the same figure. In the following we assume that we want to fit an ARIMA(p,d,q) model to z_t.

a) From the plots in Figure 1 discuss what (tentative) values you find it reasonable to use for p, d and q.

If you conclude that you want to try with a model with $d > 0$, discuss based on the plots in Figure 1 whether or not you will include a deterministic trend parameter, θ_0, in the model.

Remember to give reasons for you choices!

Independent of your answers to the above questions, assume one chooses to fit as much as six models: ARIMA(1,1,0), ARIMA(2,1,0), ARIMA(3,1,0), ARIMA(1,1,1), ARIMA(1,1,2) and ARIMA(1,1,3), all without a deterministic trend parameter. Estimated parameter values and corresponding standard deviations for the six models are given in Table 1 and plots of the estimated residuals and estimated autocorrelation functions for the estimated residuals are shown in Figure 2.

b) Based on the results in Table 1 and Figure 2, discuss briefly for each of the six ARIMA models whether or not the corresponding estimated model is a reasonable model for the observed time series.

Which of the six ARIMA models would you use for the observed time series?

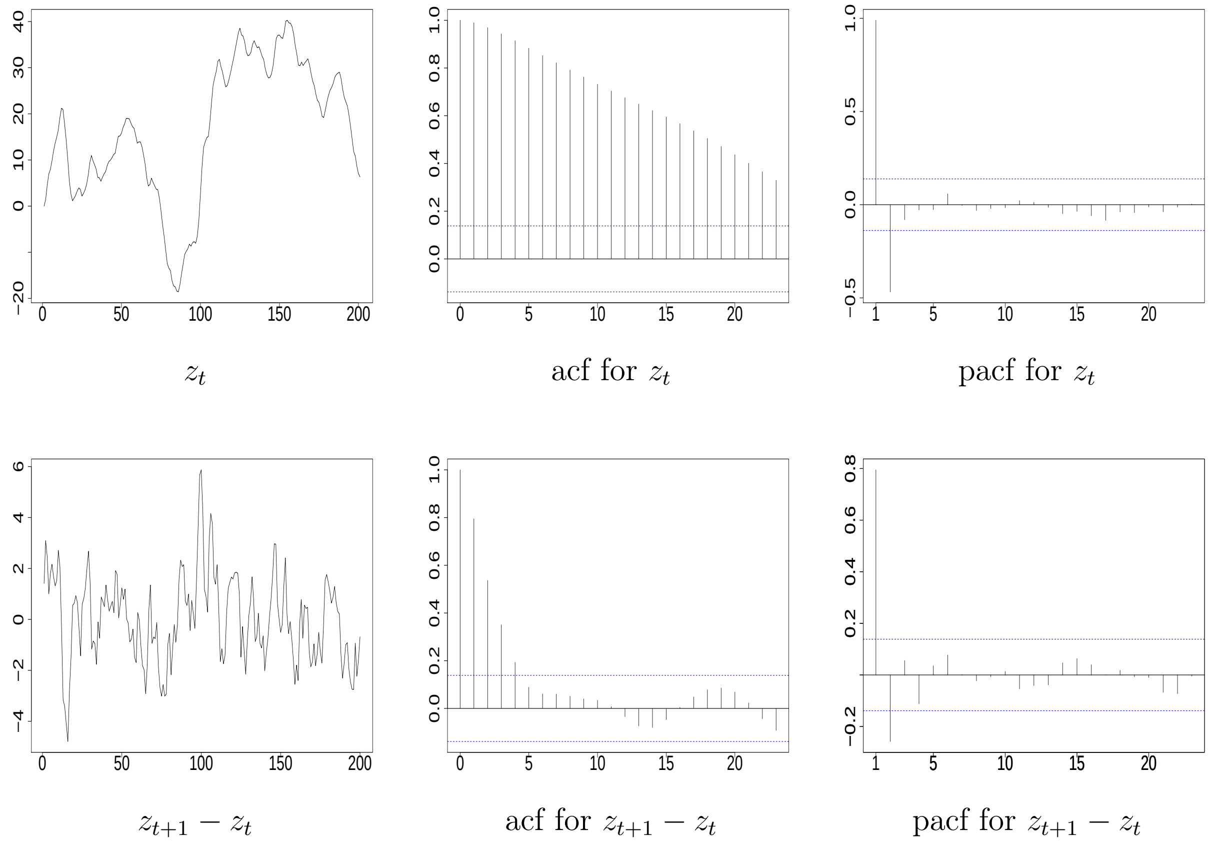

Figure 1: Upper row: Observed time series z_t and corresponding estimated autocorrelation function and partial autocorrelation function. Lower row: The differenced version of the observed time series z_t and corresponding estimated autocorrelation function and partial autocorrelation function.

	ARIMA(1,1,0)		ARIMA(2,1,0)		ARIMA(3,1,0)	
	value	s.e.	value	s.e.	value	s.e.
φ_1	0.794	0.042	0.993	0.068	1.007	0.071
φ_2			-0.251	0.069	-0.304	0.099
φ_3					0.054	0.071
σ_a^2	1.084		1.016		1.013	
log-like	-292.36		-285.94		-285.66	
AIC	588.71		577.89		579.32	

	ARIMA(1,1,1)		ARIMA(1,1,2)		ARIMA(1,1,3)	
	value	s.e.	value	s.e.	value	s.e.
φ_1	0.673	0.064	0.690	0.084	0.609	0.123
θ_1	0.350	0.084	0.330	0.107	0.398	0.130
θ_2			-0.0267	0.093	0.078	0.136
θ_3					0.105	0.101
σ_a^2	1.008		1.007		1.002	
log-like	-285.14		-285.10		-284.60	
AIC	576.29		578.20		579.21	

Table 1: Estimated parameter values with corresponding standard deviations, and optimal log-likelihood and AIC values for the six models: ARIMA(1,1,0), ARIMA(2,1,0), ARIMA(3,1,0), ARIMA(1,1,1), ARIMA(1,1,2) and ARIMA(1,1,3).

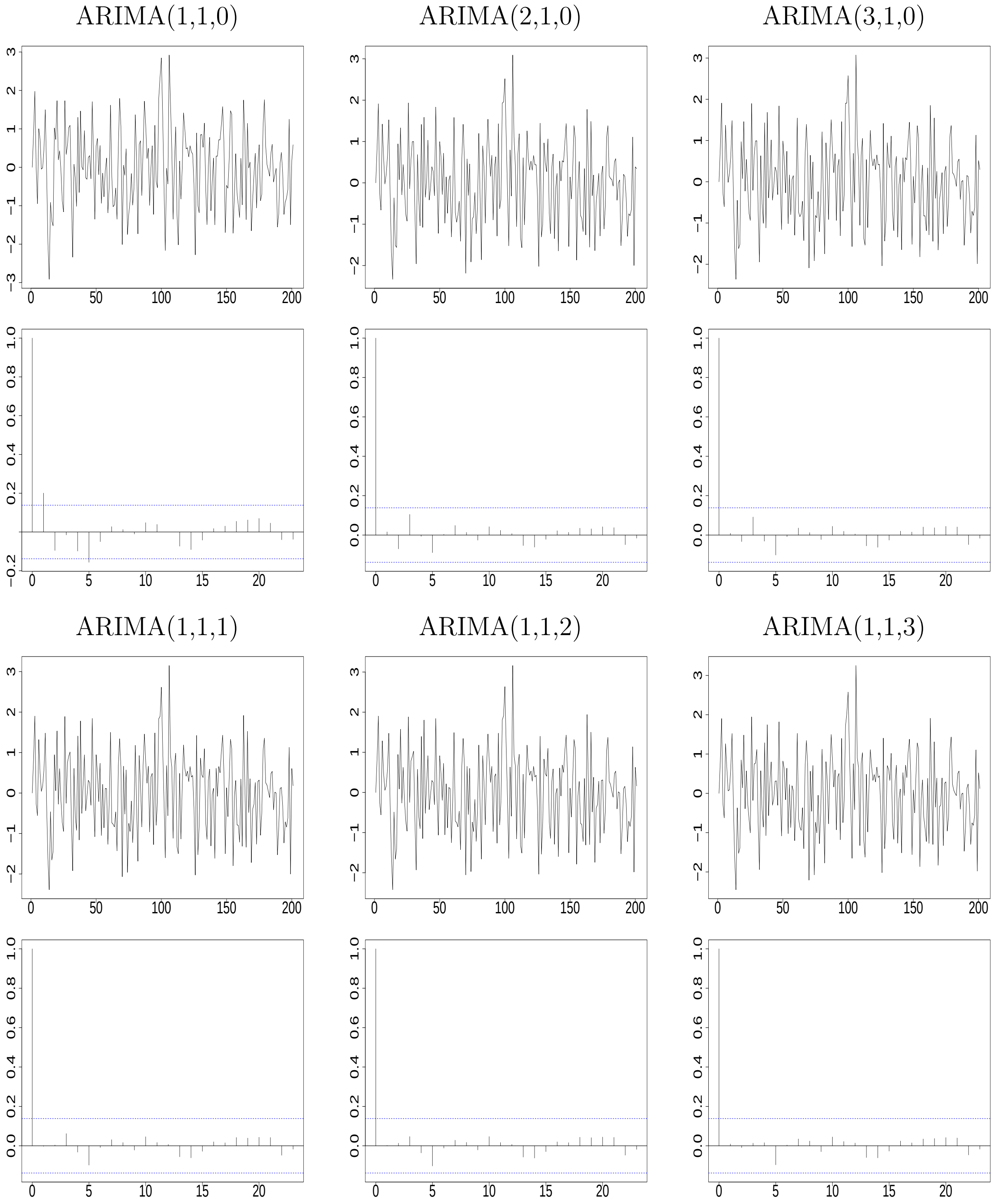

Figure 2: Estimated residuals and corresponding estimated autocorrelation functions for the six models: ARIMA(1,1,0), ARIMA(2,1,0), ARIMA(3,1,0), ARIMA(1,1,1), ARIMA(1,1,2) and ARIMA(1,1,3).

Problem 3

In this problem we will consider a state-space model in the form

$$x_{t+1} = \Phi x_t + w_{t+1} \quad \text{and} \quad y_t = A x_t + v_t \quad \text{for } t = 0, 1, \ldots, \tag{1}$$

where $x_0 \sim \mathrm{N}(\mu_0, \Sigma_0)$, $w_t \sim \mathrm{N}(0, Q)$ and $v_t \sim \mathrm{N}(0, R)$, and where x_0, $w_t, t = 1, 2, \ldots$ and $v_t, t = 0, 1, \ldots$ are all assumed to be independent.

a) Reformulate the following state-space models to the form in equation (1), i.e. for each of the two models specify the vectors and matrices x_t, y_t, Φ, A, Q and R,

1) $z_t = 0.9 z_{t-1} + a_t$, $u_t = 0.5(z_t + z_{t-1}) + b_t$,
2) $z_t = 0.9 z_{t-1} + 0.5 z_{t-2} + a_t$, $r_t = 0.5(r_{t-1} + z_{t-1}) + b_t$, $u_t = 0.5(r_t + z_t) + c_t$.

Here z_t, a_t, u_t, b_t, r_t and c_t are all scalar quantities, $\{a_t\}$, $\{b_t\}$ and $\{c_t\}$ are independent normal white-noise processes with zero mean and variances equal to σ_a^2, σ_b^2 and σ_c^2, respectively, and in both cases one only observes the process $\{u_t\}$.

In the following we will assume that we have a **scalar** state-space model in the form given by equation (1), so that x_t, y_t, Φ, w_t, Q, A, v_t, R, μ_0 and Σ_0 are all scalar.

b) Assume in this item that $\Phi = \frac{1}{2}$, $A = 2$ and $R = 1$, and that $P = \lim_{t\to\infty} P_t^t$ exists, where $P_t^t = \mathrm{Var}[x_t | y_0, \ldots, y_t]$.

Use the Kalman filter equations given in the last page of this problem set to show that

$$P = -\left(2Q + \frac{3}{8}\right) + \sqrt{Q + \left(2Q + \frac{3}{8}\right)^2}.$$

Make a sketch of P as a function of Q. Explain why the qualitative behaviour of P as a function of Q is intuitively reasonable for the state-space model considered here. In particular give an intuitive reason for the value of P when $Q = 0$.

c) For a scalar state-space model on the form in equation (1), use known properties of the multi-normal distribution to explain why the conditional distribution of x_{t+1} given $y_0, \ldots, y_t$ is a normal distribution. Correspondingly, explain why the conditional distribution for x_{t+1} given $y_0, \ldots, y_t, y_{t+1}$ is also a normal distribution.

Use the above to derive formulas for

$$x_{t+1}^{t+1} = \mathrm{E}[x_{t+1}|y_0, \ldots, y_{t+1}] \quad \text{and} \quad P_{t+1}^{t+1} = \mathrm{Var}[x_{t+1}|y_0, \ldots, y_{t+1}]$$

as a function of

$$x_{t+1}^{t} = \mathrm{E}[x_{t+1}|y_0, \ldots, y_t], \quad P_{t+1}^{t} = \mathrm{Var}[x_{t+1}|y_0, \ldots, y_t]$$

and the quantities that define the state-space model. [Note that you in this item are **not** allowed to use the general Kalman filter equations given in the last page of this problem set.]

Appendix:

The Kalman filter equations (in the notation used in the lectures):

$$
\begin{aligned}
x_{t+1}^{t} &= \Phi x_t^t \\
P_{t+1}^{t} &= \Phi P_t^t \phi^T + Q \\
x_{t+1}^{t+1} &= x_{t+1}^{t} + K_{t+1}(y_{t+1} - A_{t+1} x_{t+1}^{t}) \\
P_{t+1}^{t+1} &= [I - K_{t+1} A_{t+1}] P_{t+1}^{t} \\
K_{t+1} &= P_{t+1}^{t} A_{t+1}^{T} [A_{t+1} P_{t+1}^{t} A_{t+1}^{T} + R]^{-1}
\end{aligned}
$$