TMA4285 Time Series Models
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Solution

Oppgave 1

a) A process $\{z_t\}$ is invertible if it can be represented as an AR(∞) process,

$$z_t = \sum_{j=1}^{\infty} \pi_j z_{t-j} + a_t.$$

The process is invertible if and only if all the roots of $\theta(B) = 0$ lie outside the unit circle. We have

$$\theta(B) = 1 - \frac{1}{2}B + \frac{1}{4}B^2.$$

Thus,

$$\theta(B) = 0 \Leftrightarrow \frac{1}{4}B^2 - \frac{1}{2}B + 1 = 0 \Rightarrow B = \frac{\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - 4 \cdot \frac{1}{4}}}{2 \cdot \frac{1}{4}} = \frac{\frac{1}{2} \pm \sqrt{-\frac{3}{4}}}{\frac{1}{2}}$$

$$\Rightarrow B = 1 \pm i\sqrt{3}.$$

and

$$|B|^2 = 1^2 + \sqrt{3}^2 = 1 + 3 = 4 > 1.$$

All roots are outside the unit circle, so the model is invertible.

A process $\{z_t\}$ is covariance stationary if

1. $\mathrm{E}[z_t]$ exists and is not a function of t, and
2. $\mathrm{Cov}[z_t, z_{t+k}]$ exists and is not a function of t, only a function of k.

The process is covariance stationary if and only if all roots of $\varphi(B) = 0$ lie outside the unit circle. We have

$$\varphi(B) = 1 - \frac{1}{2}B.$$

Thus,

$$\varphi(B) = 0 \Rightarrow B = 2 \Rightarrow |B| > 1.$$

All roots lie outside the unit circle, so the model is covariance stationary.

b) We have

$$\varphi(B)z_t = \theta(B)a_t \Rightarrow z_t = \frac{\theta(B)}{\varphi(B)}a_t.$$

Thus, we must have

$$\psi(B) = \frac{\theta(B)}{\varphi(B)} \Rightarrow \varphi(B)\psi(B) = \theta(B).$$

This gives

$$(1-\varphi_1 B)(1-\psi_1 B-\psi_2 B^2-\psi_3 B^2-\ldots) = 1-\theta_1 B-\theta_2 B^2.$$

By exanding on the left hand side of this equation and setting equal coefficients in front of the same power of B, we get sequentially

$$\begin{array}{ll} B^1: & -\psi_1-\varphi_1 = -\theta_1 \Rightarrow \psi_1 = \theta_1-\varphi_1, \\ B^2: & -\psi_2+\varphi_1\psi_1 = -\theta_2 \Rightarrow \psi_2 = \theta_2+\varphi_1\psi_1 = (\theta_1-\varphi_1)\varphi_1+\theta_2, \\ B^3: & -\psi_3+\varphi_1\psi_2 = 0 \Rightarrow \psi_3 = \varphi_1\psi_2 = \varphi_1((\theta_1-\varphi_1)\varphi_1+\theta_2), \\ & \vdots \\ B^k: & -\psi_k+\varphi_1\psi_{k-1} = 0 \Rightarrow \psi_k = \varphi_1\psi_{k-1} = \ldots = \varphi_1^{k-2}((\theta_1-\varphi_1)\varphi_1+\theta_2). \end{array}$$

By noting that the formula $\psi_k = \varphi_1^{k-2}((\theta_1-\varphi_1)+\theta_2)$ is valid also for $k=2$, we have shown what is asked for.

c)

$$\begin{aligned} \gamma_0 = \text{Var}[z_t] = \text{Var}\left[\sum_{j=0}^{\infty}\psi_j a_{t-j}\right] &= \sum_{j=0}^{\infty}\psi_j^2 \text{Var}[a_{t-j}] = \sigma_a^2\left[1^2+\psi_1^2+\sum_{j=2}^{\infty}\psi_j^2\right] \\ &= \sigma_a^2\left[1+(\theta_1-\varphi_1)^2+\sum_{j=2}^{\infty}\varphi_1^{2(j-2)}((\theta_1-\varphi_1)\varphi_1+\theta_2)^2\right] \\ &= \sigma_a^2\left[1+(\theta_1-\varphi_1)^2+((\theta_1-\varphi_1)\varphi_1+\theta_2)^2\sum_{j=0}^{\infty}\varphi_1^{2j}\right] \\ \Rightarrow \gamma_0 &= \sigma_a^2\left[1+(\theta_1-\varphi_1)^2+\frac{((\theta_1-\varphi_1)\varphi_1+\theta_2)^2}{1-\varphi_1^2}\right] \end{aligned}$$

$$\gamma_1 = \text{Cov}[z_t, z_{t+1}] = \text{Cov}\left[\sum_{j=0}^{\infty}\psi_j a_{t-j}, \sum_{i=0}^{\infty}\psi_i a_{t+1-i}\right] = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\psi_j\psi_i \text{Cov}[a_{t-j}, a_{t+1-i}]$$

As we have

$$\text{Cov}[a_j, a_i] = \begin{cases} \sigma_a^2 & \text{if } t-j = t+1-i \Leftrightarrow i = j+1, \\ 0 & \text{otherwise,} \end{cases}$$

the double sum reduces to a single sum,

$$\gamma_1 = \sum_{j=0}^{\infty}\psi_j\psi_{j+1}\sigma_a^2 = \sigma_a^2\left[\psi_1+\psi_1\psi_2+\sum_{j=2}^{\infty}\psi_j\psi_{j+1}\right]$$

$$= \sigma_a^2 \left[\psi_1 + \psi_1\psi_2 + \sum_{j=2}^{\infty} \varphi_1^{j-2}\varphi_1^{j+1-2}((\theta_1 - \varphi_1)\varphi_1 + \theta_2)^2 \right]$$

$$= \sigma_a^2 \left[\psi_1(1 + \psi_2) + ((\theta_1 - \varphi_1)\varphi_1 + \theta_2)^2 \varphi_1 \sum_{j=0}^{\infty} \varphi_1^{2j} \right]$$

$$= \sigma_a^2 \left[\psi_1(1 + \psi_2) + \frac{((\theta_1 - \varphi_1)\varphi_1 + \theta_2)^2\varphi_1}{1 - \varphi_1^2} \right]$$

$$\Rightarrow \underline{\underline{\gamma_1 = \sigma_a^2 \left[(\theta_1 - \varphi_1)(1 + (\theta_1 - \varphi_1)\varphi_1 + \theta_2) + \frac{((\theta_1 - \varphi_1)\varphi_1 + \theta_2)^2\varphi_1}{1 - \varphi_1^2} \right]}}.$$

Oppgave 2

a) From the plot of $\{z_t\}$, the time series is clearly not stationary. We can see this also from the fact that $\widehat{\rho}_k$ decays very slowly and $\widehat{\phi}_{kk}$ goes quickly to zero.

The differenced time series seems to be stationary. Thus, we choose $d = 1$.

The acf for the differenced time series decays exponentially to zero, whereas the corresponding pacf appears to cut after lag 2. Thus, it seems reasonable to try an ARIMA$(2, 1, 0)$ model.

It is not natural to include a deterministic trend parameter θ_0 because the observed $\{z_t\}$ has no clear deterministic trend.

b) We first discuss the fit of each of the six proposed ARIMA models in turn.

ARIMA$(1, 1, 0)$: the acf for the estimated residuals is significantly different from zero at lag 1. Thus, this model does not give a good fit for the observed data.

ARIMA$(2, 1, 0)$: the acf for the estimated residuals looks ok, and all the estimated parameters are significant. This looks like a promising model.

ARIMA$(3, 1, 0)$: the acf for the estimated residuals looks ok, but φ_3 is not significantly different from zero. Thus, the model seems to be overparameterised.

ARIMA$(1, 1, 1)$: the acf for the estimated residuals looks ok and all estimated parameters are significant. This looks like a promising model.

ARIMA$(1, 1, 2)$: the acf for the estimated residuals looks ok, but θ_2 is not significantly different from zero. Thus, the model seems to be overparameterised.

ARIMA$(1, 1, 3)$: the acf for the estimated residuals looks ok, but θ_2 and θ_3 are not significantly different from zero. Thus, the model seems to be overparameterised.

The ARIMA$(2, 1, 0)$ and ARIMA$(1, 1, 1)$ models both give a good fit for the observed data. These models have the same number of parameters and the log-likelihood is slightly lower for the ARIMA$(1, 1, 1)$ model (and thereby also the AIC is slightly lower for the ARIMA$(1, 1, 1)$ model), so for this reason we prefer the ARIMA$(1, 1, 1)$ model.

Oppgave 3

a) 1) Define $x_t = \begin{bmatrix} z_t \\ z_{t-1} \end{bmatrix}$ and $y_t = u_t$. We then get

$$x_{t+1} = \begin{bmatrix} z_{t+1} \\ z_t \end{bmatrix} = \begin{bmatrix} 0.9 & 0 \\ 1 & 0 \end{bmatrix} x_t + \begin{bmatrix} a_t \\ 0 \end{bmatrix} \quad \text{and} \quad y_t = [u_t] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} x_t + [b_t].$$

Thus we have

$$\Phi = \begin{bmatrix} 0.9 & 0 \\ 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

$$Q = \text{Cov}\left(\begin{bmatrix} a_t \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \sigma_a^2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad R = \text{Cov}([b_t]) = \sigma_b^2.$$

2) Define $x_t = \begin{bmatrix} z_t \\ z_{t-1} \\ r_t \end{bmatrix}$ and $y_t = u_t$. We then get

$$x_{t+1} = \begin{bmatrix} z_{t+1} \\ z_t \\ r_{t+1} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.5 & 0 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix} x_t + \begin{bmatrix} a_t \\ 0 \\ b_t \end{bmatrix} \quad \text{and} \quad y_t = [u_t] = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} x_t + [c_t].$$

Thus, we have

$$\Phi = \begin{bmatrix} 0.9 & 0.5 & 0 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}, \quad A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$Q = \text{Cov}\left(\begin{bmatrix} a_t \\ 0 \\ b_t \end{bmatrix} \right) = \begin{bmatrix} \sigma_a^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_b^2 \end{bmatrix} \quad \text{and} \quad R = \text{Cov}([c_t]) = [\sigma_c^2].$$

b) Inserting values for Φ, A and R in the Kalman recursions we get

$$P_{t+1}^t = \frac{1}{4} P_t^t + Q,$$

$$P_{t+1}^{t+1} = (1 - K_{t+1} \cdot 2) P_{t+1}^t,$$

$$K_{t+1} = \frac{2P_{t+1}^t}{4P_{t+1}^t + 1}.$$

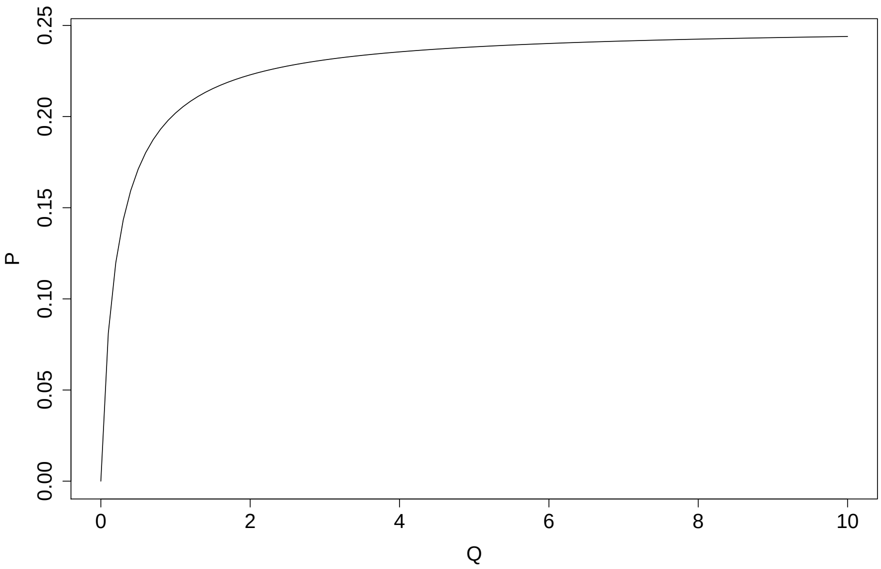
Eliminating K_{t+1} and P_{t+1}^t we obtain

$$P_{t+1}^{t+1} = \left(1 - \frac{\frac{1}{2} P_t^t + 2Q}{P_t^t + 4Q + 1} \cdot 2 \right) \left(\frac{1}{4} P_t^t + Q \right).$$

Letting $t \to \infty$ we get

$$P = \left(1 - \frac{P + 4Q}{P + 4Q + 1} \right) \left(\frac{1}{4} P + Q \right)$$

$$P(P + 4Q + 1) = \left(\frac{1}{4} P + Q \right) (P + 4Q + 1) - (P + 4Q) \left(\frac{1}{4} P + Q \right)$$



Figur 1: Plot of P as a function of Q in Problem 3**b**.

$$P(P+4Q+1) = \frac{1}{4}P + Q$$

$$P^2 + 4QP + P - \frac{1}{4}P - Q = 0$$

$$P^2 + \left(4Q + \frac{3}{4}\right) - Q = 0$$

$$P = \frac{-\left(4Q+\frac{3}{4}\right) \pm \sqrt{\left(4Q+\frac{3}{4}\right)^2 + 4Q}}{2}$$

As we must have $P \geq 0$, the root with minus in front of the square root is invalid. Thus,

$$P = \frac{-\left(4Q+\frac{3}{4}\right) + \sqrt{\left(4Q+\frac{3}{4}\right)^2 + 4Q}}{2} = -\left(2Q + \frac{3}{8}\right) + \sqrt{Q + \left(2Q + \frac{3}{8}\right)^2}.$$

Plot of P as a function of Q is given in Figure 1. When $Q = 0$ the x_t process is deterministic. Note however, that we do not know the initial value of the x_t process and our observations are noisy (since $R > 0$), so $P_t^t > 0$ for all (finite) t. When t grows we get more and more information about the (deterministic) process and in the limit when $t \to \infty$ we know exactly what it is, thus $P = 0$ in this case.

When $Q > 0$ the x_t process is random and, for any t, the available information about w_t is only y_t. Thus, the larger the w_t's tend to be, the more uncertain we are about the value of x_t. Thus, P will grow as a function of Q. In the limit when $Q \to \infty$ the x_t process is a white noise process (with a very large variance) and our only source of information about x_t at time t is y_t. As the observation noise does not change with Q, P will converge to a finite value when $Q \to \infty$.

c) We have that

$$x_{t+1} = \Phi x_t + w_{t+1} = \Phi(\Phi x_{t-1} + w_t) + w_{t+1} = \Phi^2 x_{t-1} + \Phi w_t + w_{t+1}$$
$$= \ldots = \Phi^{t+1} x_0 + \Phi^t w_1 + \ldots + \Phi w_t + w_{t+1}$$

and

$$y_s = Ax_s + v_s = A\Phi^s x_0 + A\Phi^{s-1} w_1 + \ldots + A\Phi w_{s-1} + Aw_s + v_s.$$

Thereby the vectors $[x_{t+1}\ y_0\ y_1\ \ldots\ y_t]^T$ and $[x_{t+1}\ y_0\ y_1\ \ldots\ y_{t+1}]^T$ are linear combinations of independent normally distributed random variables. Thus, the two vectors are multi-normally distributed. Then also $x_{t+1}|y_0,\ldots,y_t$ and $x_{t+1}|y_0,\ldots,y_{t+1}$ are normal.

We have

$$x_{t+1}|y_0,\ldots,y_t \sim \mathrm{N}\left(x_{t+1}^t, P_{t+1}^t\right), \tag{3.1}$$
$$x_{t+1}|y_0,\ldots,y_{t+1} \sim \mathrm{N}\left(x_{t+1}^{t+1}, P_{t+1}^{t+1}\right), \tag{3.2}$$
$$y_{t+1}|x_t \sim \mathrm{N}(Ax_{t+1}, R). \tag{3.3}$$

Using (3.1) and (3.3) we get (where proportionalities are as a function of x_{t+1})

$$f(x_t|y_0,\ldots,y_{t+1}) \propto f(x_{t+1}, y_{t+1}|y_0,\ldots,y_t)$$
$$= f(x_{t+1}|y_0,\ldots,y_t)\cdot f(y_{t+1}|x_{t+1},y_0,\ldots,y_t)$$
$$= f(x_{t+1}|y_0,\ldots,y_t)\cdot f(y_{t+1}|x_{t+1})$$
$$\propto \exp\left\{-\frac{1}{2}\frac{(x_{t+1}-x_{t+1}^t)^2}{P_{t+1}^t}\right\}\cdot\exp\left\{-\frac{1}{2}\frac{(y_{t+1}-Ax_{t+1})^2}{R}\right\}$$
$$\propto \exp\left\{-\frac{1}{2}\left[\frac{1}{P_{t+1}^t}x_{t+1}^2 - \frac{2x_{t+1}^t}{P_{t+1}^t}x_{t+1} - \frac{2Ay_{t+1}}{R}x_{t+1} + \frac{A^2}{R}x_{t+1}^2\right]\right\}.$$

From (3.2) we get (where again the proportionalities are as a function of x_{t+1})

$$f(x_{t+1}|y_0,\ldots,y_{t+1}) \propto \exp\left\{-\frac{1}{2}\frac{(x_{t+1}-x_{t+1}^{t+1})^2}{P_{t+1}^{t+1}}\right\}$$
$$\propto \exp\left\{-\frac{1}{2}\left[\frac{1}{P_{t+1}^{t+1}}x_{t+1}^2 - \frac{2x_{t+1}^{t+1}}{P_{t+1}^{t+1}}x_{t+1}\right]\right\}.$$

Thereby we must have that the coefficients in front of x_{t+1}^2 in the two expressions we have found for $f(x_{t+1}|y_0,\ldots,y_{t+1})$ must be equal, and correspondingly for the linear terms. Thus,

$$\frac{1}{P_{t+1}^{t+1}} = \frac{1}{P_{t+1}^t} + \frac{A^2}{R} \Rightarrow P_{t+1}^{t+1} = \frac{1}{\frac{1}{P_{t+1}^t} + \frac{A^2}{R}}$$
$$\Rightarrow P_{t+1}^{t+1} = \underline{\underline{\frac{P_{t+1}^t R}{R + A^2 P_{t+1}^t}}}$$

and

$$\frac{2x_{t+1}^{t+1}}{P_{t+1}^{t+1}} = \frac{2x_{t+1}^t}{P_{t+1}^t} + \frac{2Ay_{t+1}}{R} \Rightarrow \underline{\underline{x_{t+1}^{t+1} = P_{t+1}^{t+1}\left(\frac{x_{t+1}^t}{P_{t+1}^t} + \frac{Ay_{t+1}}{R}\right)}}.$$