NTNU

Department of Mathematical Sciences

Examination paper for
Solution: TMA4285 Time series models

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Examination date: December 7th 2013
Examination time (from–to): 09:00–13:00
Permitted examination support material: C:

- Calculator CITIZEN SR-270X, CITIZEN SR-270X College or HP30S.
- Statistiske tabeller og formler, Tapir forlag.
- K. Rottman: Matematisk formelsamling.
- One yellow, stamped A5 sheet with own handwritten formulas and notes.

Other information:
Note that all answers should be justified.
In your solution you can use English and/or Norwegian.

Language: English
Number of pages: 5
Number pages enclosed: 0

Checked by:

Date Signature

Problem 1

a) Both ts1 and ts2 seem to be stationary , so one should have $d = 0$ for both time series.

The acf for ts1 seems to be a damped sine wave, whereas the pacf cuts off after lag 3. This behaviour is consistent with an AR(3) model, so one should start with $\underline{\underline{p = 3,\ d = 0 \text{ and } q = 0}}$.

The acf for ts2 cuts off after lag 1, whereas the pacf seems to decay exponentially. This behaviour is consistent with an MA(1) model, so one should start with $\underline{\underline{p = 0,\ d = 0 \text{ and } q = 1}}$.

b) For the ARMA(1,1) model the estimated values for φ_1 and θ_1 are both significant, but the estimated acf and pacf have several lags clearly outside the confidence bands and this indicates that there are is correlation in the time series that can not be modelled by this model.

For the ARMA(1,2) model the estimated value for θ_2 is (barely) not significant, which may indicate over-parameterisation. The estimated acf and pacf for the estimated residuals have at least one lag clearly outside the confidence bands, which again indicates that there is some correlation in the time series that can not be modelled by this model.

For the ARMA(2,1) model the estimated values for φ_1, φ_2 and θ_1 are all significant and the estimated acf and pacf for the estimated residuals are inside the confidence bands except for a few lags where the values are slightly outside the confidence bands. Thereby this seems to be a good model for the observed time series.

From the above discussion it should be clear that one should use the ARMA(2,1) model for ts3.

Problem 2

a) For a time series z_t to be second-order stationary one must have

$$F_{z_{t_1}, z_{t_2}}(x_1, x_2) = F_{z_{t_1+k}, z_{z_2+k}}(x_1, x_2),$$

for all t_1, t_2, k and all x_1, x_2.

For a time series z_t to be covariance stationary all first and second order moments must exist and be time invariant.

A time series process z_t which is covariance stationary, does not need to be second-order stationary. Even if the two first moments is time invariant, the joint distribution does not need to be time invariant.

A time series process z_t which is second-order stationary does not need to be covariance stationary. If all the first and second order moments exist for a second-order stationary process, then it is also covariance stationary, but the first and second order moments do not need to exist for a second-order stationary process.

b) The process is invertible if and only if all the roots of $\theta(B) = 1 - \theta_1 B = 0$ are outside the unit circle. The current model has only one root, which is $B = 1/\theta_1$ so thereby the process is invertible if

$$\left|\frac{1}{\theta_1}\right| > 1 \Leftrightarrow \underline{\underline{\theta \in (-1, 1)}}.$$

The model is covariance stationary if and only if the roots of $\varphi(B) = 1 - 1.7B + 0.72B^2 = 0$ are outside the unit circle. The roots in the current model become

$$B = \frac{1.7 \pm \sqrt{1.7^2 - 4 \cdot 0.72}}{2 \cdot 0.72} = \frac{1.7 \pm \sqrt{0.01}}{1.44} \Rightarrow B = 1.25 \text{ or } B = 1.11.$$

Thus, $\underline{\underline{\text{the model is covariance stationary}}}$.

c) From the model we have that

$$z_{t+k} = \varphi_1 z_{t+k-1} + \varphi_2 z_{t+k-2} + a_{t+k} - \theta_1 a_{t+k-1}.$$

Thereby we get for $k = 0, 1, 2, \ldots$,

$$\begin{aligned}
\gamma_k &= \mathrm{E}[z_{t+k} z_t] = \mathrm{E}[(\varphi_1 z_{t+k-1} + \varphi_2 z_{t+k-2} + a_{t+k} - \theta_1 a_{t+k-1}) z_t] \\
&= \varphi_1 \mathrm{E}[z_{t+k-1} z_t] + \varphi_2 \mathrm{E}[z_{t+k-2} z_t] + \mathrm{E}[a_{t+k} z_t] - \theta_1 \mathrm{E}[a_{t+k-1} z_t] \\
&= \varphi_1 \gamma_{k-1} + \varphi_2 \gamma_{k-2} + \mathrm{E}[a_{t+k} z_t] - \theta_1 \mathrm{E}[a_{t+k-1} z_t].
\end{aligned}$$

For $k = 2, 3, \ldots$ we have that both $t + k$ and $t + k - 1$ is larger that t and thereby a_{t+k} and a_{t+k-1} are both uncorrelated with z_t, so $\mathrm{E}[a_{t+k} z_t] = \mathrm{E}[a_{t+k}]\mathrm{E}[z_t] = 0 \cdot \mathrm{E}[z_t] = 0$ and $\mathrm{E}[a_{t+k-1} z_t] = \mathrm{E}[a_{t+k-1}]\mathrm{E}[z_t] = 0 \cdot \mathrm{E}[z_t] = 0$. Thus we have found the homogeneous difference equation

$$\underline{\underline{\gamma_k - \varphi_1 \gamma_{k-1} - \varphi_2 \gamma_{k-2} = 0 \quad \text{for } k = 2, 3, \ldots}} \qquad (1)$$

To find the initial conditions we need to consider the above expression for γ_k for $k < 2$. We start with $k = 0$,

$$\begin{aligned}
\gamma_0 &= \varphi_1\gamma_{-1} + \varphi_2\gamma_{-2} + \mathrm{E}[a_t z_t] - \theta_1\mathrm{E}[a_{t-1}z_t] \\
&= \varphi_1\gamma_{-1} + \varphi_2\gamma_{-2} + \mathrm{E}[a_t(\varphi_1 z_{t-1} + \varphi_2 z_{t-2} + a_t - \theta_1 a_{t-1})] \\
&\quad -\theta_1\mathrm{E}[a_{t-1}(\varphi_1 z_{t-1} + \varphi_2 z_{t-2} + a_t - \theta_1 a_{t-1})] \\
&= \varphi_1\gamma_{-1} + \varphi_2\gamma_{-2} + \mathrm{E}[a_t \cdot a_t] - \theta_1(\varphi_1\mathrm{E}[a_{t-1}z_{t-1}] - \theta_1\mathrm{E}[a_{t-1}\cdot a_{t-1}]) \\
&= \varphi_1\gamma_{-1} + \varphi_2\gamma_{-2} + \sigma_a^2 - \theta_1(\varphi_1\sigma_a^2 - \theta_1\sigma_a^2).
\end{aligned}$$

Thus, using that γ_k is a symmetric function, the first equation becomes

$$\gamma_0 = \varphi_1\gamma_1 + \varphi_2\gamma_2 + \sigma_a^2(1 - \theta_1\varphi_1 + \theta_1^2). \tag{2}$$

Doing the same for $k = 1$ we get

$$\gamma_1 = \varphi_1\gamma_0 + \varphi_2\gamma_{-1} + \mathrm{E}[a_{t+1}z_t] - \theta_1\mathrm{E}[a_t z_t].$$

Again using what we found above, namely that $\mathrm{E}[a_{t+1}z_t] = 0$ and $\mathrm{E}[a_t z_t] = \sigma_a^2$ the second equation becomes

$$\gamma_1 = \varphi_1\gamma_0 + \varphi_2\gamma_1 - \theta_1\sigma_a^2. \tag{3}$$

As (2) and (3) includes both γ_0, γ_1 and γ_2 we need a third equation, which we get for $k = 2$. This, however, we get directly from (1),

$$\gamma_2 = \varphi_1\gamma_1 + \varphi_2\gamma_0. \tag{4}$$

Thereby we have three equations, (2), (3) and (4), with three unknowns, γ_0, γ_1 and γ_2, which can be solved to find the initial conditions.

d) We have

$$\varphi(B)z_t = \theta(B)a_t \Rightarrow \frac{\varphi(B)}{\theta(B)}z_t = a_t.$$

Thereby we must have

$$\psi(B) = \frac{\varphi(B)}{\theta(B)} \Rightarrow \theta(B)\psi(B) = \varphi(B),$$

where $\psi(B) = 1 - \psi_1 B - \psi_2 B^2 - \ldots$. Thereby,

$$(1 - \theta_1 B)(1 - \psi_1 B - \psi_2 B^2 - \psi_3 B^3 - \ldots) = 1 - \varphi_1 B - \varphi_2 B^2.$$

By expanding on the left hand side of this equation and setting equal coefficients in front of the same power of B, we sequentially get

$$\begin{aligned}
B^1: &\quad -\psi_1 - \theta_1 = -\varphi_1 \Rightarrow = \psi_1 = \varphi_1 - \theta_1, \\
B^2: &\quad -\psi_2 + \theta_1\psi_1 = -\varphi_2 \Rightarrow \psi_2 = \varphi_2 + \theta_1\psi_1 = \varphi_2 + \theta_1(\varphi_1 - \theta_1), \\
B^3: &\quad -\psi_3 + \theta_1\psi_2 = 0 \Rightarrow \psi_3 = \theta_1\psi_2 = \theta_1(\varphi_2 + \theta_1(\varphi_1 - \theta_1)), \\
&\quad \vdots \\
B^k: &\quad -\psi_k + \theta_1\psi_{k-1} = 0 \Rightarrow \psi_k = \theta_1\psi_{k-1} = \theta_1^{k-2}(\varphi_2 + \theta_1(\varphi_1 - \theta_1)).
\end{aligned}$$

e) The one-step ahead forecast can be found directly from the AR(∞) representation

$$\begin{aligned}
\widehat{z}_n(1) &= \mathrm{E}[z_{n+1}| \ldots, z_{n-1}, z_n] = \mathrm{E}\left[\sum_{k=1}^{\infty} \psi_k z_{n+1-k} + a_{n+1} \middle| \ldots, z_{n-1}, z_n\right] \\
&= \sum_{k=1}^{\infty} \psi_k z_{n+1-k},
\end{aligned}$$

where we have used that a_{n+1} is uncorrelated with the observed values and that $\mathrm{E}[a_{n+1}] = 0$.

From the definition of the model we get for $k \geq 2$

$$\begin{aligned}
\widehat{z}_n(k) &= \mathrm{E}[z_{n+k}| \ldots, z_{n-1}, z_n] \\
&= \mathrm{E}[\varphi_1 z_{n+k-1} + \varphi_2 \mathrm{E}[z_{n+k-2} + a_{n+k} - \theta_1 a_{n+k-1}| \ldots, z_{n-1}, z_n] \\
&= \varphi_1 \mathrm{E}[z_{n+k-1}| \ldots, z_{n-1}, z_n] + \varphi_2 \mathrm{E}[z_{n+k-2}| \ldots, z_{n-1}, z_n] \\
&= \varphi_1 \widehat{z}_n(k-1) + \varphi_2 \widehat{z}_n(k-2),
\end{aligned}$$

where we have used that a_{n+k} and a_{n+k-1} are uncorrelated with the observed data when $k \geq 2$. Thereby we have the homogeneous difference equation (inserting given parameter values)

$$\widehat{z}_n(k) - 1.7\widehat{z}_n(k-1) + 0.72\widehat{z}_n(k-2) = 0 \quad \text{for } k = 2, 3, \ldots.$$

To write up the general solution of this difference equation we need to find the roots of $1 - 1.7B + 0.72B^2 = 0$. These, however, we have previously found to be $B = 1.25 = 1/0.8$ and $B = 1.11 = 1/0.9$. The general solution is thereby

$$\widehat{z}_n(k) = b_0 \cdot 0.8^k + b_1 0.9^k.$$

We find b_0 and b_1 from the initial conditions $\widehat{z}_n(1)$ and $\widehat{z}_n(0) = z_n$,

$$\begin{aligned}
b_0 \cdot 0.8 + b_1 0.9 &= \widehat{z}_n(1), \\
b_0 \cdot 1 + b_1 \cdot 1 &= z_n,
\end{aligned}$$

which have solution $b_0 = 9z_n - 10\widehat{z}_n(1)$ and $b_1 = 10\widehat{z}_n(1) - 8z_n$. Thereby the eventual forecast function is

$$\underline{\underline{\widehat{z}_n(k) = (9z_n - 10\widehat{z}_n(1)) \cdot 0.8^k + (10\widehat{z}_n(1) - 8z_n) \cdot 0.9^k}}.$$

Problem 3

a) We use the notation $x_t^n = \mathrm{E}[x_t|y_0, \ldots, y_n]$ and $P_t^n = \mathrm{Cov}[x_t|y_0, \ldots, y_n]$. From the model assumptions we then get for $k = 1, 2, \ldots$

$$\begin{aligned} x_{n+k}^n &= \mathrm{E}[x_{n+k}|y_0, \ldots, y_n] = \mathrm{E}[\Phi x_{n+k-1} + w_{n+k}|y_0, \ldots, y_n] \\ &= \Phi \mathrm{E}[x_{n+k-1}|y_0, \ldots, y_n] + \mathrm{E}[w_{n+k}|y_0, \ldots, y_n] \\ &= \Phi x_{n+k-1}^n + \mathrm{E}[w_{n+k}|y_0, \ldots, y_n] \end{aligned}$$

Using that w_{n+k} is independent of $y_0, \ldots, y_n$ we get $\mathrm{E}[w_{n+k}|y_0, \ldots, y_n] = \mathrm{E}[w_{n+k}] = 0$. Thus, we have the recursion

$$\underline{\underline{x_{n+k}^n = \Phi x_{n+k-1}^n \quad \text{for } k = 1, 2, \ldots}}.$$

For P_{n+k}^n we correspondingly get

$$\begin{aligned} P_{n+k}^n &= \mathrm{Cov}[x_{n+k}|y_0, \ldots, y_n] = \mathrm{Cov}[\Phi x_{n+k-1} + w_{n+k}|y_0, \ldots, y_n] \\ &= \Phi \mathrm{Cov}[x_{n+k-1}|y_0, \ldots, y_n]\Phi^T + \mathrm{E}[w_{n+k}|y_0, \ldots, y_n] \\ &= \Phi P_{n+k-1}^n \Phi^T + Q, \end{aligned}$$

where we have used that w_{n+k} is independent of both $y_0, \ldots, y_n$ and x_{n+k-1}. Thus, the recursion of the covariance matrix is

$$\underline{\underline{P_{n+k}^n = \Phi P_{n+k-1}^n \Phi^T + Q \quad \text{for } k = 1, 2, \ldots}}.$$

All $x_{n+k}, y_0, \ldots, y_n$ can be expressed as a linear combination of the independent Gaussian variables x_0, $\omega_1, \ldots, \omega_{n+k}$ and $v_0, \ldots, v_n$. Thereby the joint distribution of $x_{n+k}, y_0, \ldots, y_n$ is Gaussian, and since the Gaussian distribution is closed under conditioning, the conditional distribution of x_{n+k} given $y_0, \ldots, y_n$ is also Gaussian. If the x_t's are scalar quantities, a 95% prediction interval for x_{n+k} is

$$\underline{\underline{\left[x_{n+k}^n - 1.96\sqrt{P_{n+k}^n}, x_{n+k}^n + 1.96\sqrt{P_{n+k}^n}\right]}}.$$

If the x_t's are vectors, corresponding prediction intervals can be constructed for each component of x_{n+k}.