NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4285 Time series models**

Academic contact during examination: Håkon Tjelmeland

Phone: 4822 1896

Examination date: December 4th 2014

Examination time (from–to): 09:00–13:00

Permitted examination support material: C:

- Calculator Casio fx-82ES PLUS, Citizen SR-270X, Citizen SR-270X College or HP30S.
- Statistiske tabeller og formler, Tapir forlag.
- K. Rottman: Matematisk formelsamling.
- One yellow, stamped A5 sheet with own handwritten formulas and notes.

Other information:
Note that all answers should be justified.
In your solution you can use English and/or Norwegian.

Language: English

Number of pages: 4

Number pages enclosed: 0

Checked by:

Date Signature

Problem 1

A plot of an observed time series ts is shown in the upper left corner of Figure 1. The same figure also includes plots of the estimated autocorrelation and partial autocorrelation functions based on ts. The ts after differencing and corresponding estimated autocorrelation and partial autocorrelation functions for the differenced time series are given in the lower row of the same figure. In the following we assume we want to fit an ARIMA(p, d, q) model to ts.

Based on the plots in Figure 1, discuss what (tentative) values you find it reasonable to use for p, d and q.

If you conclude that you want to try with a model with $d > 0$, discuss based on the plots in Figure 1 whether or not you will include a deterministic trend parameter θ_0 in the model.

Conclude by writing up an equation for your suggested model.

Remember to give reasons for all your choices!

Problem 2

Consider a zero mean AR(2) process

$$z_t = \varphi_1 z_{t-1} + \varphi_2 z_{t-2} + a_t, \tag{1}$$

where $\{a_t\}_{t=-\infty}^{\infty}$ is assumed to be white noise with $\mathrm{E}[a_t] = 0$ and $\mathrm{Var}[a_t] = \sigma_a^2$. For this process we first consider the following three parameter sets.

i) $\varphi_1 = 1$ and $\varphi_2 = -\frac{1}{4}$

ii) $\varphi_1 = -\frac{3}{2}$ and $\varphi_2 = -\frac{9}{8}$

iii) $\varphi_1 = \frac{3}{2}$ and $\varphi_2 = -\frac{13}{16}$

The AR(2) process is stationary for two of these three parameter sets.

a) Find for which two parameter sets the process is stationary. Give reasons for your answer.

Figure 2 shows the autocorrelation functions for the AR(2) process for the parameter sets specified above where the process is stationary.

b) Decide which autocorrelation function belongs to which parameter set. Remember to give reasons for your answers.

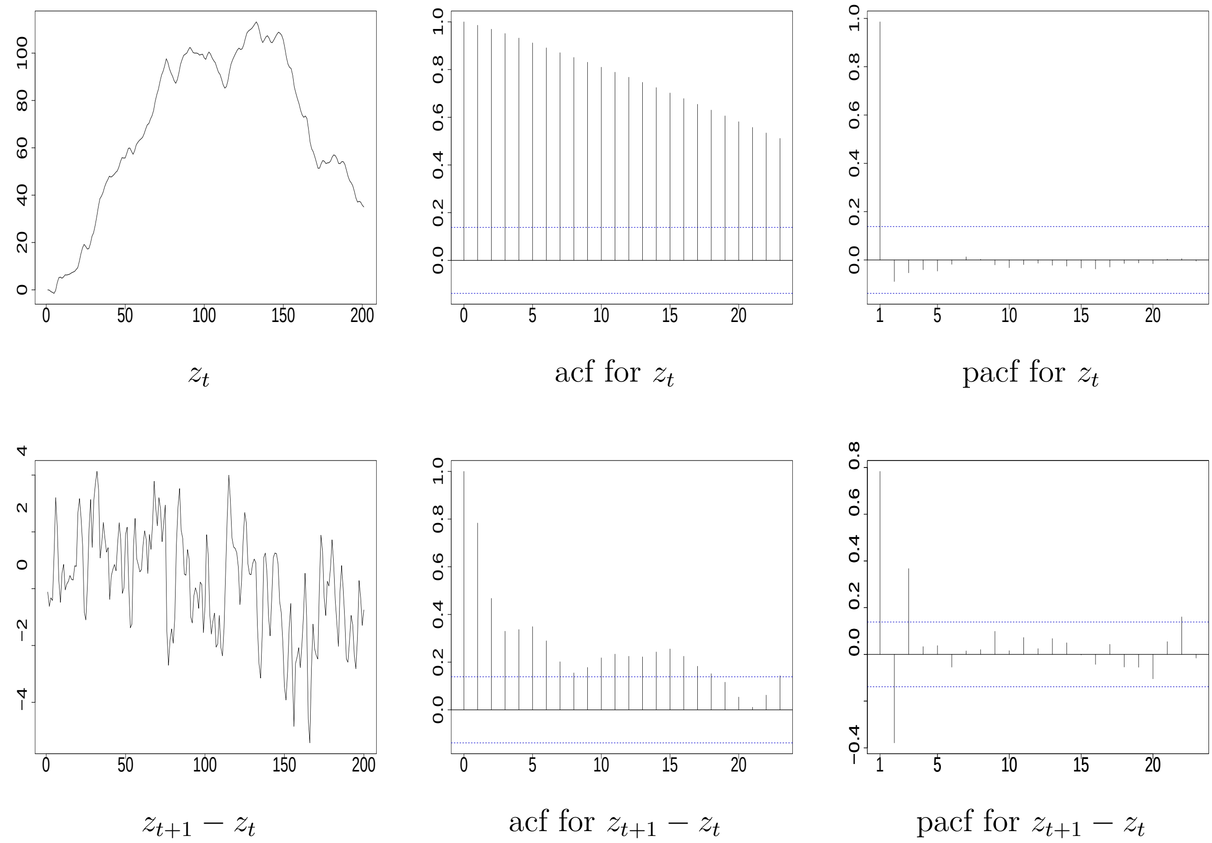
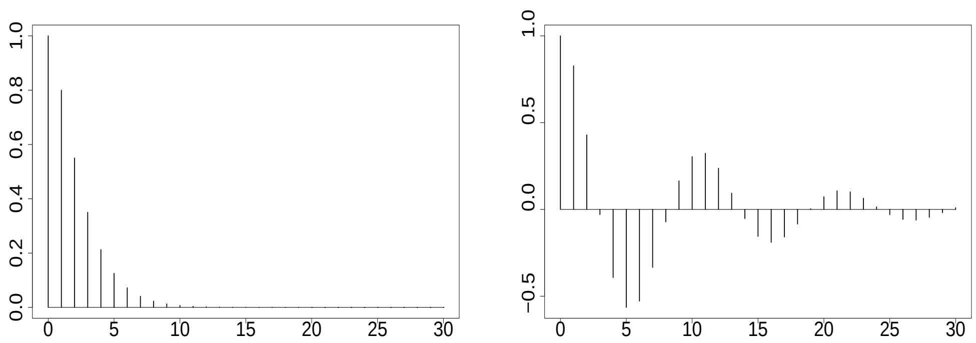


Figure 1: Upper row: Observed time series ts and corresponding estimated autocorrelation and partial autocorrelation functions. Lower row: The differenced version of ts and corresponding estimated autocorrelation and partial autocorrelation functions.



(a) (b)

Figure 2: Autocorrelation functions for two stationary AR(2) processes.

In the following we assume the parameters φ_1 and φ_2 to have values so that the AR(2) process is stationary. The AR(2) process can then be represented as an MA(∞) process,

$$z_t = \psi(B)a_t \quad \text{where} \quad \psi(B) = 1 + \sum_{k=1}^{\infty} \psi_k B^k.$$

c) Find expressions for the coefficients ψ_1 and ψ_2 as functions of φ_1 and φ_2 and develop a homogeneous difference equation for the remaining ψ_k coefficients.

For $\varphi_1 = \frac{7}{6}$ and $\varphi_2 = -\frac{1}{3}$, solve the homogeneous difference equation and show that

$$\psi_k = 4 \cdot \left(\frac{2}{3}\right)^k - 3 \cdot \left(\frac{1}{2}\right)^k \quad \text{for} \quad k = 1, 2, \ldots$$

In the following we continue to focus on the AR(2) process when $\varphi_1 = \frac{7}{6}$ and $\varphi_2 = -\frac{1}{3}$, and in addition we assume $\text{Var}[a_t] = \sigma_a^2 = 1$. We will consider forecasting for such a process. Assume we are at time n and have observed the values $\{z_t\}_{t=-\infty}^{n}$ and in particular the two last observed values are $z_{n-1} = -0.89$ and $z_n = 0.20$. We want to forecast z_{n+l} for $l = 1, 2, \ldots$ and find the variance of the associated forecast error. To forecast z_{n+l} we use

$$\widehat{z}_n(l) = \text{E}[z_{n+l}|z_t, t \leq n]$$

and the associated forecast error is $e_n(l) = z_{n+l} - \widehat{z}_n(l)$.

d) Find values for the one- and two-step ahead forecasts $\widehat{z}_n(1)$ and $\widehat{z}_n(2)$.

Find also values for the variances of the one- and two step ahead forecast errors, $\text{Var}[e_n(1)]$ and $\text{Var}[e_n(2)]$.

e) Develop a general formula for the l-step ahead forecast, $\widehat{z}_n(l)$. Note: Your formula should be a function of l only.

Develop also a general formula for the associated forecast error, $\text{Var}[e_n(l)]$. Hint: Start from the MA(∞) representation found above.

Find in particular the limits $\lim_{l\to\infty} \widehat{z}_n(l)$ and $\lim_{l\to\infty} \text{Var}[e_n(l)]$ and explain these limiting values.

Problem 3

A state-space model can be formulated in the form

$$x_{t+1} = \Phi x_t + \omega_{t+1} \quad \text{and} \quad y_t = A x_t + v_t \quad \text{for } t = 0, 1, \ldots,$$

where $x_0 \sim \text{N}(\mu_0, \Sigma_0)$, $\omega_t \sim \text{N}(0, Q)$ and $v_t \sim \text{N}(0, R)$, and where $x_0, \omega_t, t = 1, 2, \ldots$ and $v_t, t = 0, 1, \ldots$ are all assumed to be independent.

Defining $x_t^s = \text{E}[x_t | y_0, \ldots, y_s]$ and $P_t^s = \text{Cov}[x_t | y_0, \ldots, y_s]$ the forecast recursions for the state-space model is given by

$$x_{n+l}^n = \Phi x_{n+l-1}^n \quad \text{and} \quad P_{n+l}^n = \Phi P_{n+l-1}^n \Phi^T + Q,$$

for $l = 1, 2, \ldots$

Reformulate the AR(2) model in (1) in the state-space form above, i.e. specify the vectors x_t, y_t, ω_t and v_t and the matrices Φ, A, Q and R. Hint: Let the state vector consist of z_t and z_{t-1}.

As in the last part of Problem 2, now assume $\varphi_1 = \frac{7}{6}$ and $\varphi_2 = -\frac{1}{3}$ and that the two last observed values are $z_{n-1} = -0.89$ and $z_n = 0.20$.

Use the state-space formulation of the AR(2) model and the forecast recursions given above to find values for the one- and two-step forecasts. Find also the associated forecast variances using the forecast recursions given above. Compare your results here with your results in Problem 2**d**) and discuss.