NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for
Solution: TMA4285 Time series models

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Examination time (from–to): 09:00–13:00
Permitted examination support material: C:

- Calculator Casio fx-82ES PLUS, Citizen SR-270X, Citizen SR-270X College or HP30S.
- Statistiske tabeller og formler, Tapir forlag.
- K. Rottman: Matematisk formelsamling.
- One yellow, stamped A5 sheet with own handwritten formulas and notes.

Other information:
Note that all answers should be justified.
In your solution you can use English and/or Norwegian.

Language: English
Number of pages: 7
Number pages enclosed: 0

Checked by:

Date Signature

Problem 1 The time series is clearly non-stationary in the mean, so we need $d > 0$. We see this both from the time series itself and from the slowly decaying ACF and behaviour of the PACF. The differenced time series seems stationary in the mean, which then imply $d = 1$. The ACF for the differenced time series seems to decay exponentially, and the PACF seems to cut off after lag 3. This is consistent with a model with $p = 3$ and $q = 0$. Thus an ARIMA$(3, 1, 0)$ model.

From the plot of the time series we see that there is not at all a clear trend, so there is no reason to include the deterministic trend parameter θ_0.

The model thus becomes

$$\left(1 - \varphi_1 B - \varphi_2 B^2 - \varphi_3 B^3\right)(1 - B)\, z_t = a_t$$

Problem 2

a) To check whether the AR(2) model is stationary we need to find the roots of

$$1 - \varphi_1 B - \varphi_2 B^2 = 0,$$

or following standard practice, the roots of

$$\varphi_2 B^2 + \varphi_1 B - 1 = 0,$$

$$B = \frac{-\varphi_1 \pm \sqrt{\varphi_1^2 + 4\varphi_2}}{2\varphi_2}.$$

The process is stationary if both roots are outside the unit circle.

i) $\varphi_1 = 1$ and $\varphi_2 = -\frac{1}{4}$:

$$B_1 = \frac{-1 + \sqrt{1 - 1}}{2\left(-\frac{1}{4}\right)} = \frac{4}{2} = 2$$

$$B_2 = \frac{-1 - \sqrt{0}}{2\left(-\frac{1}{4}\right)} = 2$$

Thus, both roots are outside the unit circle and the process is stationary.

ii) $\varphi_1 = -\frac{3}{2}$ and $\varphi_2 = -\frac{9}{8}$:

$$B_1 = \frac{\frac{3}{2} + \sqrt{\left(-\frac{3}{2}\right)^2 - 4 \cdot \frac{9}{8}}}{2\left(-\frac{9}{8}\right)} = \frac{\frac{3}{2} + \sqrt{\frac{9}{4} - \frac{9}{2}}}{-\frac{9}{4}} = \frac{\frac{3}{2} + \sqrt{-\frac{9}{4}}}{-\frac{9}{4}}$$
$$= \frac{\frac{3}{2} + i\frac{3}{2}}{-\frac{9}{4}} = -\frac{2}{3} - i\frac{2}{3}$$
$$|B_1|^2 = \left(-\tfrac{2}{3}\right)^2 + \left(-\tfrac{2}{3}\right)^2 = \frac{4}{9} + \frac{4}{9} = \frac{8}{9} < 1$$

Thus, at least one of the roots is not outside the unit circle and the process is not stationary.

iii) $\varphi_1 = \frac{3}{2}$ and $\varphi_2 = -\frac{13}{16}$:

$$B_1 = \frac{-\frac{3}{2} + \sqrt{\left(\frac{3}{2}\right)^2 - 4 \cdot \frac{13}{16}}}{2\left(-\frac{13}{16}\right)} = \frac{-\frac{3}{2} + \sqrt{\frac{9}{4} - \frac{13}{4}}}{-\frac{13}{8}}$$
$$= \frac{-\frac{3}{2} + \sqrt{-1}}{-\frac{13}{8}} = \frac{12}{13} - i\frac{8}{13}$$
$$|B_1|^2 = \left(\tfrac{12}{13}\right)^2 + \left(-\tfrac{8}{13}\right)^2 = \frac{12^2 + 8^2}{13^2} = 1.23 > 1$$

Since complex roots comes in conjugate pairs we then know that $B_2 = \frac{12}{13} + i\frac{8}{13}$ and $|B|^2 > 1$. Thus, both roots are outside the unit circle and the process is stationary.

b) Model *i*) has real roots, which gives an exponentially decaying ACF.

Model *iii*) has complex roots, which gives an ACF that is a damped sine wave.

Thus Model *i*) corresponds to Figure 2(a), and Model *iii*) corresponds to Figure 2(b).

c) We have

$$\varphi(B)z_t = a_t \Rightarrow z_t = \frac{1}{\varphi(B)}a_t = \psi(B)a_t \Rightarrow \varphi(B)\psi(B) = 1$$

where

$$\varphi(B) = 1 - \varphi_1 B - \varphi_2 B^2$$

Thus

$$\left(1 - \varphi_1 B - \varphi_2 B^2\right)\left(1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \dots\right) = 1$$

By collecting factors in front of equal powers of B we get:

$$B^1 : \psi_1 - \varphi_1 = 0 \Rightarrow \underline{\underline{\psi_1 = \varphi_1}}$$
$$B^2 : \psi_2 - \varphi_1\psi_1 - \varphi_2 = 0 \Rightarrow \underline{\underline{\psi_2 = \varphi_2 + \varphi_1^2}}$$

For $k \geq 3$ we get

$$-\varphi_2\psi_{k-2} - \varphi_1\psi_{k-1} + \psi_k = 0$$

So the difference equation becomes:

$$\underline{\underline{\psi_k - \varphi_1\psi_{k-1} - \varphi_2\psi_{k-2} = 0\,,\ k = 3, 4, \dots}}$$

With $\varphi_1 = \frac{7}{6}$, $\varphi_2 = -\frac{1}{3}$ the difference equation becomes

$$\psi_k - \tfrac{7}{6}\psi_{k-1} + \tfrac{1}{3}\psi_{k-2} = 0\,,\ k = 3, 4, \dots$$

so we need to find the roots of

$$\begin{aligned}
\tfrac{1}{3}B^2 - \tfrac{7}{6}B + 1 &= 0 \\
B &= \frac{\frac{7}{6} \pm \sqrt{\left(-\frac{7}{6}\right)^2 - 4\cdot\frac{1}{3}}}{2\cdot\frac{1}{3}} \\
&= \frac{\frac{7}{6} \pm \sqrt{\frac{49}{36} - \frac{48}{36}}}{\frac{2}{3}} \\
&= \frac{\frac{7}{6} \pm \frac{1}{6}}{\frac{2}{3}} \\
&= \begin{cases} \frac{\frac{8}{6}}{\frac{2}{3}} = \frac{24}{12} = 2 \\ \frac{1}{\frac{2}{3}} = \frac{3}{2} \end{cases}
\end{aligned}$$

So $B_1^{-1} = \frac{1}{2}$, $B_2^{-1} = \frac{2}{3}$ and the general solution is

$$\psi_k = b_0\left(\tfrac{1}{2}\right)^k + b_1\left(\tfrac{2}{3}\right)^k\,,\ k = 1, 2, 3, \dots$$

Initial conditions are $\psi_1 = \varphi_1 = \frac{7}{6}$ and $\psi_2 = \varphi_2 + \varphi_1^2 = \frac{37}{36}$. Thereby

$$\psi_1 = b_0\tfrac{1}{2} + b_1\tfrac{2}{3} = \tfrac{7}{6} \Rightarrow b_0 = -\tfrac{4}{3}b_1 + \tfrac{7}{3}$$
$$\psi_2 = b_0\tfrac{1}{4} + b_1\tfrac{4}{9} = \tfrac{37}{36}$$

Inserting b_0 into the equation for ψ_2 gives $b_1 = 4$ which then gives the result

$$\underline{\underline{b_0 = -3,\ b_1 = 4}}$$

d) The model is

$$z_t = \tfrac{7}{6} z_{t-1} - \tfrac{1}{3} z_{t-2} + a_t$$

$$\begin{aligned}
\widehat{z}_n(1) &= \mathrm{E}\left[z_{n+1} \,|\, z_t, t \le n\right] \\
&= \mathrm{E}\left[\tfrac{7}{6} z_n - \tfrac{1}{3} z_{n-1} + a_{n+1} \,\middle|\, z_t, t \le n\right] \\
&= \tfrac{7}{6} z_n - \tfrac{1}{3} z_{n-1} = \tfrac{7}{6} \cdot 0.20 - \tfrac{1}{3}(-0.89) \\
\Rightarrow \quad & \underline{\underline{\widehat{z}_n(1) = 0.53}} \\
\widehat{z}_n(2) &= \mathrm{E}\left[z_{n+2} \,|\, z_t, t \le n\right] \\
&= \mathrm{E}\left[\tfrac{7}{6} z_{n+1} - \tfrac{1}{3} z_n + a_{n+2} \,\middle|\, z_t, t \le n\right] \\
&= \tfrac{7}{6} \widehat{z}_n(1) - \tfrac{1}{3} z_n = \tfrac{7}{6} \cdot 0.53 - \tfrac{1}{3} \cdot 0.20 \\
\Rightarrow \quad & \underline{\underline{\widehat{z}_n(2) = 0.552}} \\
e_n(1) &= \left(\tfrac{7}{6} z_n - \tfrac{1}{3} z_{n-1} + a_{n+1}\right) - \left(\tfrac{7}{6} z_n - \tfrac{1}{3} z_{n-1}\right) = a_{n+1} \\
\mathrm{Var}\left[e_n(1)\right] &= \mathrm{Var}\left[a_{n+1}\right] = \sigma_a^2 = \underline{\underline{1}} \\
e_n(2) &= \left(\tfrac{7}{6} z_{n+1} - \tfrac{1}{3} z_n + a_{n+2}\right) - \left(\tfrac{7}{6} \widehat{z}_n(1) - \tfrac{1}{3} z_n\right) \\
&= \tfrac{7}{6}\left(z_{n+1} - \widehat{z}_n(1)\right) + a_{n+2} = \tfrac{7}{6} a_{n+1} + a_{n+2} \\
\mathrm{Var}\left[e_n(2)\right] &= \left(\left(\tfrac{7}{6}\right)^2 + 1\right) \sigma_a^2 = \frac{49 + 36}{36} \sigma_a^2 = \underline{\underline{2.36}}
\end{aligned}$$

e) For $l \ge 3$ we have

$$\begin{aligned}
\widehat{z}_n(l) &= \mathrm{E}\left[\tfrac{7}{6} z_{n+l-1} - \tfrac{1}{3} z_{n+l-2} + a_{n+l} \,\middle|\, z_t,\, t \le n\right] \\
&= \tfrac{7}{6} \widehat{z}_n(l-1) - \tfrac{1}{3} \widehat{z}_n(l-2)
\end{aligned}$$

Thus, we have the difference equation

$$\widehat{z}_n(l) - \tfrac{7}{6} \widehat{z}_n(l-1) + \tfrac{1}{3} \widehat{z}_n(l-2) = 0$$

The coefficients in this difference equation is the same as in Problem 2**c**), so the roots are

$$\frac{1}{B_1} = \frac{1}{2}, \quad \frac{1}{B_2} = \frac{2}{3}$$

and the general solution is

$$\widehat{z}_n(l) = b_0 \left(\tfrac{1}{2}\right)^l + b_1 \left(\tfrac{2}{3}\right)^l, \quad l = 1, 2, 3, \ldots$$

with initial conditions

$$\begin{aligned}
\widehat{z}_n(1) &= b_0 \cdot \tfrac{1}{2} + b_1 \cdot \tfrac{2}{3} = 0.53 \\
\widehat{z}_n(2) &= b_0 \cdot \tfrac{1}{4} + b_1 \cdot \tfrac{4}{9} = 0.552
\end{aligned}$$

Solving these equations together with respect to b_0 and b_1 gives

$$b_0 = -2.384\,,\ b_1 = 2.583$$

so

$$\underline{\underline{\widehat{z}_n(l) = -2.384\left(\tfrac{1}{2}\right)^l + 2.583\left(\tfrac{2}{3}\right)^l\ ,\ l = 1, 2, \ldots}}$$

For the associated forecast error, we have

$$z_t = \sum_{k=0}^{\infty} \psi_k a_{t-k}$$

and hence

$$\begin{aligned}
\widehat{z}_n(l) &= \mathrm{E}\left[z_{n+l} \,|\, z_t,\, t \le n\right] \\
&= \mathrm{E}\left[\sum_{k=0}^{\infty} \psi_k a_{n+l-k} \,\middle|\, z_t,\, t \le n\right] = \sum_{k=0}^{\infty} \psi_k \mathrm{E}\left[a_{n+l-k} \,|\, z_t,\, t \le n\right] \\
&= \sum_{k=0}^{l-1} \psi_k \underbrace{\mathrm{E}\left[a_{n+l-k} \,|\, z_t,\, t \le n\right]}_{=0} + \sum_{k=l}^{\infty} \psi_k \underbrace{\mathrm{E}\left[a_{n+l-k} \,|\, z_t,\, t \le n\right]}_{=a_{n+l-k}} \\
&= \sum_{k=l}^{\infty} \psi_k a_{n+l-k}
\end{aligned}$$

Consequently

$$\begin{aligned}
e_n(l) &= z_{n+l} - \widehat{z}_n(l) = \sum_{k=0}^{l-1} \psi_k a_{n+l-k} \\
\mathrm{Var}\left[e_n(l)\right] &= \sum_{k=0}^{l-1} \psi_k^2 \sigma_a^2 = 1 + \sum_{k=1}^{l-1} \left(3\left(\frac{1}{2}\right)^k - 4\left(\frac{2}{3}\right)^k\right)^2 \\
&= 1 + \sum_{k=1}^{l-1} \left[9\left(\frac{1}{4}\right)^k - 2 \cdot 3 \cdot 4\left(\frac{1}{2} \cdot \frac{2}{3}\right)^k + 16\left(\frac{4}{9}\right)^k\right] \\
&= 1 + \frac{9}{4}\sum_{k=0}^{l-2}\left(\frac{1}{4}\right)^k - 24 \cdot \frac{1}{3}\sum_{k=0}^{l-2}\left(\frac{1}{3}\right)^k + 16 \cdot \frac{4}{9}\sum_{k=0}^{l-2}\left(\frac{4}{9}\right)^k \\
&= 1 + \frac{9}{4} \cdot \frac{1 - \left(\frac{1}{4}\right)^{l-1}}{1 - \frac{1}{4}} - 8\frac{1 - \left(\frac{1}{3}\right)^{l-1}}{1 - \frac{1}{3}} + \frac{64}{9} \cdot \frac{1 - \left(\frac{4}{9}\right)^{l-1}}{1 - \frac{4}{9}} \\
&= \underline{\underline{1 + 3\left(1 - \left(\frac{1}{4}\right)^{l-1}\right) - 12\left(1 - \left(\frac{1}{3}\right)^{l-1}\right) + \frac{64}{5}\left(1 - \left(\frac{4}{9}\right)^{l-1}\right)}}
\end{aligned}$$

The limits of $\widehat{z}_n(l)$ and $\mathrm{Var}\left[e_n(l)\right]$ are

$$\lim_{l\to\infty} \widehat{z}_n(l) = \mathrm{E}\left[z\right] = \underline{\underline{0}}$$
$$\lim_{l\to\infty} \mathrm{Var}\left[e_n(l)\right] = \underline{\underline{\frac{24}{5}}}$$

To intuitively understand these values one can note that

$$\lim_{l\to\infty} \widehat{z}_n(l) = \mathrm{E}[z_t] = 0 \quad \text{and} \quad \lim_{l\to\infty} \mathrm{Var}[e_n(l)] = \gamma_0 = \mathrm{Var}[z_t],$$

where the last can be found by solving

$$\gamma_0 = \mathrm{Var}[z_t] = \mathrm{E}[z_t^2] = \mathrm{E}\left[\left(\frac{7}{6}z_{t-1} - \frac{1}{3}z_{t-2} + a_t\right)^2\right] = \left(\frac{7}{6}\right)^2\gamma_0 + \frac{1}{3^2}\gamma_0 - 2\cdot\frac{7}{6}\cdot\frac{1}{3}\gamma_1 + 1$$

together with

$$\gamma_1 = \mathrm{Cov}[z_{t-1}, z_t] = \mathrm{E}[z_{t-1}z_t] = \mathrm{E}\left[z_{t-1}\left(\frac{7}{6}z_{t-1}z_{t-1} - \frac{1}{3}z_{t-2} + a_t\right)\right] = \frac{7}{6}\gamma_0 - \frac{1}{3}\gamma_1,$$

which gives $\gamma_0 = \frac{24}{5}$ and $\gamma_1 = \frac{21}{5}$.

Problem 3 Let $x_t = \begin{bmatrix} z_t & z_{t-1}\end{bmatrix}^T$. We then get

$$\begin{bmatrix} z_t \\ z_{t-1}\end{bmatrix} = \begin{bmatrix} \varphi_1 & \varphi_2 \\ 1 & 0\end{bmatrix}\begin{bmatrix} z_{t-1} \\ z_{t-2}\end{bmatrix} + \begin{bmatrix} a_t \\ 0\end{bmatrix}$$

so

$$\Phi = \begin{bmatrix} \varphi_1 & \varphi_2 \\ 1 & 0\end{bmatrix}, \; \omega_t = \begin{bmatrix} a_t \\ 0\end{bmatrix}, \; Q = \mathrm{Cov}\left[w_t\right] = \begin{bmatrix} \sigma_a^2 & 0 \\ 0 & 0\end{bmatrix}$$

The observed value at time t is z_t so we get

$$y_t = z_t = \begin{bmatrix} 1 & 0\end{bmatrix}\begin{bmatrix} z_t \\ z_{t-1}\end{bmatrix} + 0$$

so $A = \begin{bmatrix} 0 & 1\end{bmatrix}$ and $R = \mathrm{Var}\left[0\right] = 0$. Moreover, we have

$$x_n^n = \mathrm{E}\left[x_n \,|\, y_t,\, t \le n\right] = \mathrm{E}\left[\begin{bmatrix} z_n \\ z_{n-1}\end{bmatrix}\,\middle|\, z_t,\, t \le n\right] = \begin{bmatrix} z_n \\ z_{n-1}\end{bmatrix}$$
$$P_n^n = \mathrm{Cov}\left[x_n \,|\, y_t,\, t \le n\right] = \mathrm{Cov}\left[\begin{bmatrix} z_n \\ z_{n-1}\end{bmatrix}\,\middle|\, z_t,\, t \le n\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}$$

Using the first forecast recursions we get

$$
\begin{aligned}
x_{n+1}^{n} &= \begin{bmatrix} \varphi_1 & \varphi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_n \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} \varphi_1 z_n + \varphi_2 z_{n-1} \\ z_n \end{bmatrix} = \begin{bmatrix} 0.53 \\ 0.20 \end{bmatrix} \\
x_{n+2}^{n} &= \begin{bmatrix} \varphi_1 & \varphi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \varphi_1 z_n + \varphi_2 z_{n-1} \\ z_n \end{bmatrix} = \begin{bmatrix} \varphi_1 \left(\varphi_1 z_n + \varphi_2 z_{n-1}\right) + \varphi_2 z_n \\ \varphi_1 z_n + \varphi_2 z_{n-1} \end{bmatrix} = \begin{bmatrix} 0.552 \\ 0.53 \end{bmatrix} \\
P_{n+1}^{n} &= \Phi P_n^n \Phi^T + Q = Q = \begin{bmatrix} \sigma_a^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
P_{n+2}^{n} &= \Phi P_{n+1}^n \Phi^T + Q = \begin{bmatrix} \varphi_1 & \varphi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varphi_1 & 1 \\ \varphi_2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \varphi_1^2 + 1 & \varphi_1 \\ \varphi_1 & 1 \end{bmatrix} = \begin{bmatrix} 2.36 & \frac{7}{6} \\ \frac{7}{6} & 1 \end{bmatrix}
\end{aligned}
$$

We see that that we get the same results as in Problem 2**d**):

$$
\begin{aligned}
\left(x_{n+1}^{n}\right)_1 &= \widehat{z}_n(1) \\
\left(x_{n+2}^{n}\right)_1 &= \widehat{z}_n(2) \\
\left(P_{n+1}^{n}\right)_{11} &= \operatorname{Var}\left[e_n(1)\right] \\
\left(P_{n+2}^{n}\right)_{11} &= \operatorname{Var}\left[e_n(2)\right]
\end{aligned}
$$

which is as it should be as we have the same model and the same observed data.