NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **TMA4285 Time Series Models**

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Examination date: December 16, 2015

Examination time (from–to): 09:00–13:00

Permitted examination support material: *Tabeller og formler i statistikk*, Tapir Forlag, K. Rottmann: *Matematisk formelsamling*, Calculator Casio fx-82ES PLUS, CITIZEN SR-270X, CITIZEN SR-270X College or HP30S, one yellow A5-sheet with your own handwritten notes.

Other information:
Note that you should explain your reasoning behind your answers. You may write in English and/or Norwegian. You may write with a pencil.

Language: English

Number of pages: 3

Number of pages enclosed: 1

Checked by:

Date Signature

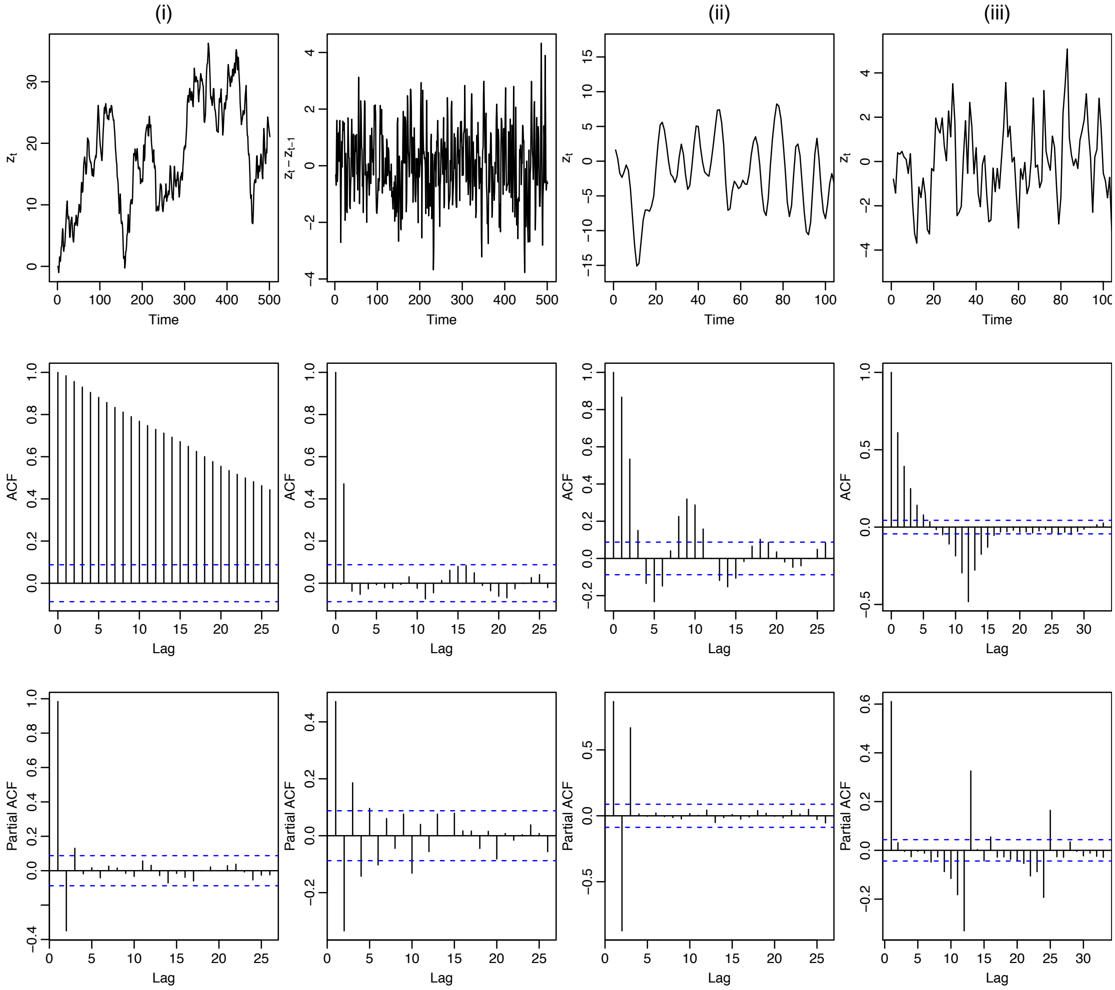

Figure 1: Three different times series (i), (ii) and (iii) (first row) and their associated sample autocorrelation (second row) and sample partial autocorrelation functions (third row). The second column shows series (i) after first order differencing.

Problem 1

a) Choose the order of ARIMA(p, d, q) models or seasonal ARIMA$(p, d, q) \times (P, D, Q)_s$ models if you were to fit such models to datasets (i), (ii) and (iii) in Fig. 1.

b) For series (i), find method of moment estimates of the autoregressive and/or moving average parameters of your chosen model that agrees with the estimated sample autocorrelation of $z_t - z_{t-1}$ at lag 1, $\hat{\rho}_1 = 0.472$. Explain which parameter values are preferable if there are several solutions.

Problem 2

Consider the ARMA(1,1) model

$$(1 - \phi_1 B)Z_t = (1 - \theta_1 B)a_t \tag{1}$$

where B is the backshift operator and a_t is zero-mean white noise with variance σ_a^2.

a) Find the coefficients of the pure AR(∞)-representation of the model. For which values of θ_1 does this representation exist?

Suppose that the model parameters are $\theta_1 = 0.4$, $\phi_1 = 0.9$ and $\sigma_a^2 = 4$ and that our last 15 observations up to time $t = 15$ are
$0.72, 3.1, 1.79, 2.32, 1.84, 0.9, 1.3, -0.71, 1.54, 0.96, 2.97, 2.28, 1.15, 1.93, 0.56$.

b) Compute, to a decimal accuracy of 3 digits, the 1-step ahead forecast $\hat{Z}_{15}(1)$ of Z_{16} based on the assumption that observations of Z_t for all $t \leq 15$ are available.

c) Derive a general formula for the l-step ahead forecast given a forecast time origin of $t = 15$. Give an interpretation of the limiting value of the forecast as $l \to \infty$.

d) Write the model in pure MA(∞) form. Use this to derive the variance of the l-step ahead forecast error $Z_{15+l} - \hat{Z}_{15}(l)$. Give an interpretation of the limiting value of this variance as $l \to \infty$.

Problem 3

Suppose that $\{Z_t\}_{t=-\infty}^{\infty}$ follow a zero-mean AR(1)-process

$$Z_t = \phi_1 Z_{t-1} + a_t \tag{2}$$

where a_t is zero-mean white noise with variance σ_a^2.

a) Derive the conditional and the exact likelihood function for this model given $n = 5$ observations $Z_1, Z_2, \ldots, Z_5$. Hint: Make use of the probability chain rule.

Suppose that the observed values are $2.0, 0.8, 0.2, -0.2, 0.1$ and that we know that $\phi_1 = 0.5$.

b) Find the conditional and the exact maximum likelihood estimate of σ_a^2. Briefly discuss why there is a difference. In particular, is the observed value of Z_1 a likely outcome given the model and given that σ_a^2 equals the conditional MLE?

Problem 4

Consider the state space model specified by the state equation

$$Y_{t+1} = AY_t + a_{t+1}, \tag{3}$$

where $A = 1/2$ and a_t is zero-mean white noise with unit variance, and the observation equation

$$Z_t = Y_t. \tag{4}$$

All quantities are scalars. Here we will assume that only $Z_1 = 2$ and $Z_5 = 2$ have been observed and that the intermediate observations Z_2, Z_3 and Z_4 are missing. Letting $\hat{Y}_{t|s}$ and $V_{t|s}$ as usual denote $E(Y_t|Z_k, k \leq s)$ and $\mathrm{Var}(Y_t|Z_k, k \leq s)$ respectively, the recursion equations in the appendix (page i) hold.

a) Explain how the filtering recursions change in cases where observations are missing. Compute $\hat{Y}_{t|t}$, $V_{t|t}$ and $\hat{Y}_{t+1|t}$ and $V_{t+1|t}$ for $t = 1, 2, \ldots, 5$.

b) Next compute $\hat{Y}_{t|5}$ and $V_{t|5}$ for $t = 1, 2, \ldots, 5$. Draw a graph visualizing the estimated states and their uncertainty and briefly discuss if the results appear reasonable.

Appendix: The Kalman recursions:

Model:

$$\begin{aligned} Y_{t+1} &= AY_t + Ga_{t+1}, & a_t &\sim N(0,\Sigma) \\ Z_t &= HY_t + b_t, & b_t &\sim N(0,\Omega) \end{aligned}$$

Forecasting:

$$\begin{aligned} \hat{Y}_{t+1|s} &= A\hat{Y}_{t|s}, \\ V_{t+1|s} &= AV_{t|s}A^\top + G\Sigma G^\top. \end{aligned}$$

Filtering:

$$\begin{aligned} \hat{Y}_{t+1|t+1} &= \hat{Y}_{t+1|t} + K_{t+1}(Z_{t+1} - H\hat{Y}_{t+1|t}), \\ V_{t+1|t+1} &= (I - K_{t+1}H)V_{t+1|t}, \\ K_{t+1} &= V_{t+1|t}H^\top(HV_{t+1|t}H^\top + \Omega)^{-1}. \end{aligned}$$

Smoothing:

$$\begin{aligned} \hat{Y}_{t|n} &= \hat{Y}_{t|t} + J_t(\hat{Y}_{t+1|n} - \hat{Y}_{t+1|t}), \\ V_{t|n} &= V_{t|t} + J_t(V_{t+1|n} - V_{t+1|t})J_t^\top, \\ J_t &= V_{t|t}A^\top V_{t+1|t}^{-1}. \end{aligned}$$