TMA4285 December 2015 Time series models, solution.

Problem 1

a) (i) The slow decay of the ACF of z_t suggest that the model is non-stationary. The differenced series and its ACF appears stationary which suggests that $d = 1$. The ACF of $(1 - B)z_t$ cuts off after lag $k = 1$ whereas the PACF exhibit exponential decay with alternating signs which suggest that $(1 - B)z_t$ is MA(1). Overall the model is then ARIMA(0,1,1).

(ii) The PACF cuts of after lag $k = 3$ whereas the ACF exhibits damped occilations aswell as possibly an exponentially decaying component consistent with an AR(3) model with two complex and one positive real root (any complex roots necessarily appear in conjungate pairs so at least one root needs to be real). So this is an ARIMA(3,0,0) model.

(iii) The large autocorrelation at lag $k = 12$ suggest a seasonal model with period $s = 12$. The overall sample ACF can be seen as a convolution between the ACF of a regular AR(1) model (the ACF decays exponentially and the PACF at cuts off at lag $k = 2$) and the ACF of MA(1) model at multiples of the seasonal period s (the ACF at lag $k = 2s = 24$ is zero). Thus the observed pattern appears consistent with a seasonal ARIMA$(1, 0, 0) \times (0, 0, 1)_{12}$ model.

b) Given that $w_t = (1 - B)z_t$ is MA(1), the two first autocovariances are $\gamma_0 = \sigma_a^2(1 + \theta_1^2)$ and $\gamma_1 = -\sigma_a^2\theta_1$. The theoretical autocorrelation at lag 1 is thus $\rho_1 = -\theta_1/(1 + \theta_1^2)$. Equating this to the sample autocorrelation $\hat{\rho}_1$ and solving for θ_1 yields two solutions

$$\theta_1 = -1.41, \qquad \theta_1 = -0.71$$

out of which we choose $\theta_1 = -0.71$ as our estimate to satisfy the requirement of invertibility.

Problem 2

a) To write

$$(1 - \phi_1 B)Z_t = (1 - \theta_1 B)a_t$$

in pure AR(∞) form

$$\pi(B)Z_t = a_t,$$

$\pi(B)$ must satisfy

$$(1 - \pi_1 B - \pi_2 B^2 - \ldots)(1 - \theta_1 B) = (1 - \phi_1 B)$$

Expanding and equating like terms yields $\pi_1 = \phi_1 - \theta_1$ and $\pi_j = \theta_1\pi_{j-1} = \theta_1^{j-1}(\phi_1 - \theta_1)$ for $j \geq 2$. This AR(∞) representation only exist when the MA part of the model is invertible for $|\theta_1| < 1$.

b) Assuming that 0.56 is the last observation Z_{15}, the 1-step ahead forecast becomes

$$\begin{aligned}\hat{Z}_{15}(1) &= E(Z_{16}|Z_{15}, Z_{14}, \ldots)\\ &= \pi_1 Z_{15} + \pi_2 Z_{14} + \ldots\\ &= (\phi_1 - \theta_1)(Z_{15} + \theta_1 Z_{14} + \theta_1^2 Z_{13} + \ldots)\\ &= .5 \cdot (.56 + .4 \cdot 1.93 + .16 \cdot 1.15 + .064 \cdot 2.28 + .0256 \cdot 2.97 + .0102 \cdot .96 + .0041 \cdot 1.54 + .0016 \cdot (-.71)\ldots)\\ &= 0.876.\end{aligned}$$

c) For lead times $l \geq 2$, the forecasts become

$$\begin{aligned}\hat{Z}_{15}(l) &= E(Z_{15+l}|Z_{15}, \ldots)\\ &= E(\phi_1 Z_{15+l-1} + a_{15+l} - \theta_1 a_{15+l-1}|Z_{15}, \ldots)\\ &= \phi_1 E(Z_{15+l-1}|Z_{15}, \ldots)\\ &= \phi_1 \hat{Z}_{15}(l-1)\\ &= \phi_1^{l-1}\hat{Z}_{15}(1)\\ &= .9^{l-1} 0.876.\end{aligned}$$

As $l \to \infty$ this tends to the stationary mean of the process, $E(Z_t) = 0$.

d) The $MA(\infty)$ polynomial $\psi(B) = \theta(B)/\phi(B)$ must satisfy

$$(1 + \psi_1 B + \psi_2 B^2 + \ldots)(1 - \phi_1 B) = (1 - \theta_1 B)$$

Expanding and equating like terms yields $\psi_1 = \phi_1 - \theta_1$ and $\psi_j = \phi_1 \psi_{j-1}$ for $j \geq 2$. Hence $\psi_j = \phi_1^{j-1}(\phi_1 - \theta_1)$ for $j \geq 1$.
The variance of the l-step ahead forecast error is then

$$\begin{aligned}\mathrm{Var}(Z_{t+l} - \hat{Z}_t(l)) &= \mathrm{Var}(\sum_{j=0}^{l-1} \psi_j a_{t+l-j})\\ &= \sigma_a^2 \sum_{j=0}^{l-1} \psi_j^2\\ &= \sigma_a^2 \left(1 + (\phi_1 - \theta_1)^2 \sum_{j=1}^{l-1} \phi_1^{2(j-1)}\right)\\ &= \sigma_a^2 \left(1 + (\phi_1 - \theta_1)^2 \sum_{j=0}^{l-2} (\phi_1^2)^j\right)\\ &= \sigma_a^2 \left(1 + (\phi_1 - \theta_1)^2 \frac{1 - \phi_1^{2(l-1)}}{1 - \phi_1^2}\right).\end{aligned}$$

As $l \to \infty$, this tends to

$$\sigma_a^2 \left(1 + \frac{(\phi_1 - \theta_1)^2}{1 - \phi_1^2}\right) = \sigma_a^2 \frac{1 + \theta_1^2 - 2\phi_1\theta_1}{1 - \phi_1^2} = \mathrm{Var}(Z_t),$$

the stationary variance of the process (see p. 61 in Wei).

Problem 3

a) Treating z_1 as given, the conditional likelihood takes the form

$$\begin{aligned} L^*(\phi_1, \sigma_a^2) &= f(z_2, z_3, z_4, z_5|z_1) \\ &= \prod_{t=2}^{5} f(z_t|z_{t-1}) \\ &= \prod_{t=2}^{5} \frac{1}{(2\pi\sigma_a^2)^{1/2}} e^{-\frac{(z_t-\phi_1 z_{t-1})^2}{2\sigma_a^2}} \\ &= \frac{1}{(2\pi\sigma_a^2)^{4/2}} e^{-\frac{1}{2\sigma_a^2}\sum_{t=2}^{5}(z_t-\phi_1 z_{t-1})^2}. \end{aligned}$$

Assuming that the process is stationary ($|\phi_1| < 1$), $z_1 \sim N(0, \frac{\sigma_a^2}{1-\phi_1^2})$. The exact likelihood is then

$$\begin{aligned} L(\phi_1, \sigma_a^2) &= f(z_1, z_2, z_3, z_4, z_5) \\ &= f(z_1) \prod_{t=2}^{5} f(z_t|z_{t-1}) \\ &= \frac{(1-\phi_1^2)^{1/2}}{(2\pi\sigma_a^2)^{1/2}} e^{-\frac{(1-\phi_1^2)z_1^2}{2\sigma_a^2}} \prod_{t=2}^{5} \frac{1}{(2\pi\sigma_a^2)^{1/2}} e^{-\frac{(z_t-\phi_1 z_{t-1})^2}{2\sigma_a^2}} \\ &= \frac{(1-\phi_1^2)^{1/2}}{(2\pi\sigma_a^2)^{5/2}} e^{-\frac{1}{2\sigma_a^2}[(1-\phi_1^2)z_1^2 + \sum_{t=2}^{5}(z_t-\phi_1 z_{t-1})^2]}. \end{aligned}$$

b) The corresponding log likelihoods are

$$\ln L^*(\phi_1, \sigma_a^2) = C - \frac{4}{2}\ln\sigma_a^2 - \frac{1}{2\sigma_a^2}\sum_{t=2}^{5}(z_t - \phi_1 z_{t-1})^2$$

and

$$\ln L(\phi_1, \sigma_a^2) = C - \frac{5}{2}\ln\sigma_a^2 + \frac{1}{2}\ln(1-\phi_1^2) - \frac{1}{2\sigma_a^2}[(1-\phi_1^2)z_1^2 + \sum_{t=2}^{5}(z_t - \phi_1 z_{t-1})^2].$$

For $\phi_1 = .5$, these are maximised for

$$\hat{\sigma}_a^{2*} = \frac{1}{4}\sum_{t=2}^{5}(z_t - \phi_1 z_{t-1})^2 = \frac{1}{4}(.2^2 + .2^2 + .3^2 + .2^2) = 0.0525,$$

and

$$\hat{\sigma}_a^2 = \frac{1}{5}[(1-\phi_1^2)z_1^2 + \sum_{t=2}^{5}(z_t - \phi_1 z_{t-1})^2] = \frac{1}{5}((1-.5^2)\cdot 2^2 + .2^2 + .2^2 + .3^2 + .2^2) = 0.642,$$

respectively.

The large difference is a result of the fact that z_1 contains information which is ignored when using the approximate conditional maximum likelihood approach. The conditional MLE of 0.0525 is not very consistent with the full data set, in particular, this value of σ_a^2 would imply a stationary variance $\sigma_a^2/(1-\phi_1^2) = 0.07$ (and a stationary standard deviation of 0.26) for which $z_1 \geq 2$ would be an extremely unlikely event with a probability of the order of 10^{-14}.

Table 1: Results from applying the Kalman filter

t	$\hat{Y}_{t\|t}$	$V_{t\|t}$	$\hat{Y}_{t+1\|t}$	$V_{t+1\|t}$	$\hat{Y}_{t\|5}$	$V_{t\|5}$
1	2	0	1	1	2	0
2	1	1	.5	1.25	1.1765	0.9882
3	.5	1.25	.25	1.3125	0.9412	1.1765
4	.25	1.3125	.125	1.3281	1.1765	0.9882
5	2	0			2	0

Problem 4

a) With no observations at $t = 2, 3, 4$, the filtering steps at these time points become

$$\hat{Y}_{t|t} = \hat{Y}_{t|t-1}, \qquad V_{t|t} = V_{t|t-1}, \tag{1}$$

that is, the probability distribution of Y_t conditional on observations up to time t is the same as the distribution conditional on observations up to time $t-1$. This can also be seen from the general formula for the filtering step by letting Ω (or rather the time dependent Ω_t) go to infinity which is equivalent to a missing observation. This leads to a kalman gain $K_t = 0$ from which (1) follows.

Also note that the state Y_1 and Y_5 is observed without any error, such that $\hat{Y}_{t|t} = Z_t = 2$ and $V_{t|t} = 0$ for $t = 1$ and $t = 5$ and so no initial values are needed.

Forecasting Y_2, we get $\hat{Y}_{2|1} = \phi_1 \hat{Y}_{1|1} = 0.5 \cdot 2 = 1$ and $V_{2|1} = \phi_1^2 V_{1|1} + \sigma_a^2 = 1$. Similar calculations and (1) leads to the numbers in Table 1.

b) Applying the smoothing recursions, using $\hat{Y}_{5|5} = 2$ and $V_{5|5} = 0$ as initial values, we first obtain, for $t = 4$, $J_4 = V_{4|4}\phi_1/V_{5|4} = 0.5 \cdot 1.3125/1.3281 = 0.4941$, $\hat{Y}_{4|5} = \hat{Y}_{4|4} + J_4(\hat{Y}_{5|5} - \hat{Y}_{5|4}) = .25 + 0.4941(2-0.125) = 1.1764$, $V_{4|5} = V_{4|4} + J_t^2(V_{5|5} - V_{5|4}) = 1.3125 + 0.4941^2(0-1.3281) = 0.9882$. Similar calculations leads to the numbers in Table 1.

Fig. 1 shows the estimates and their associated uncertainty. The symmetry of the autocovariance function of a stationary process Y_t and the pattern of the observed values translates to the estimated states being symmetric about $t = 3$. Additionallly, the expected value and the variances of the estimated states tends towards the stationary mean and variance of the process with increasing distance from the observations (towards the midpoint $t = 3$) as expected.

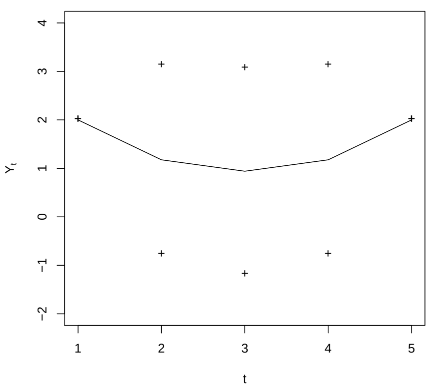

Figure 1: Expected value and 95% probability limits of $Y_t|Z_1, Z_5$.