NTNU
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Department of Mathematical Sciences

Examination paper for **TMA4285 Time Series Models**

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Examination date: December 8, 2016

Examination time (from–to): 09:00–13:00

Permitted examination support material: *Tabeller og formler i statistikk*, Tapir Forlag, K. Rottmann: *Matematisk formelsamling*, Calculator Casio fx-82ES PLUS, CITIZEN SR-270X, CITIZEN SR-270X College or HP30S, one yellow A5-sheet with your own handwritten notes.

Other information:
Note that you should explain your reasoning behind your answers. You may write in English and/or Norwegian. You may write with a pencil.

Language: English

Number of pages: 4

Number of pages enclosed: 1

Checked by:

Informasjon om trykking av eksamensoppgave	
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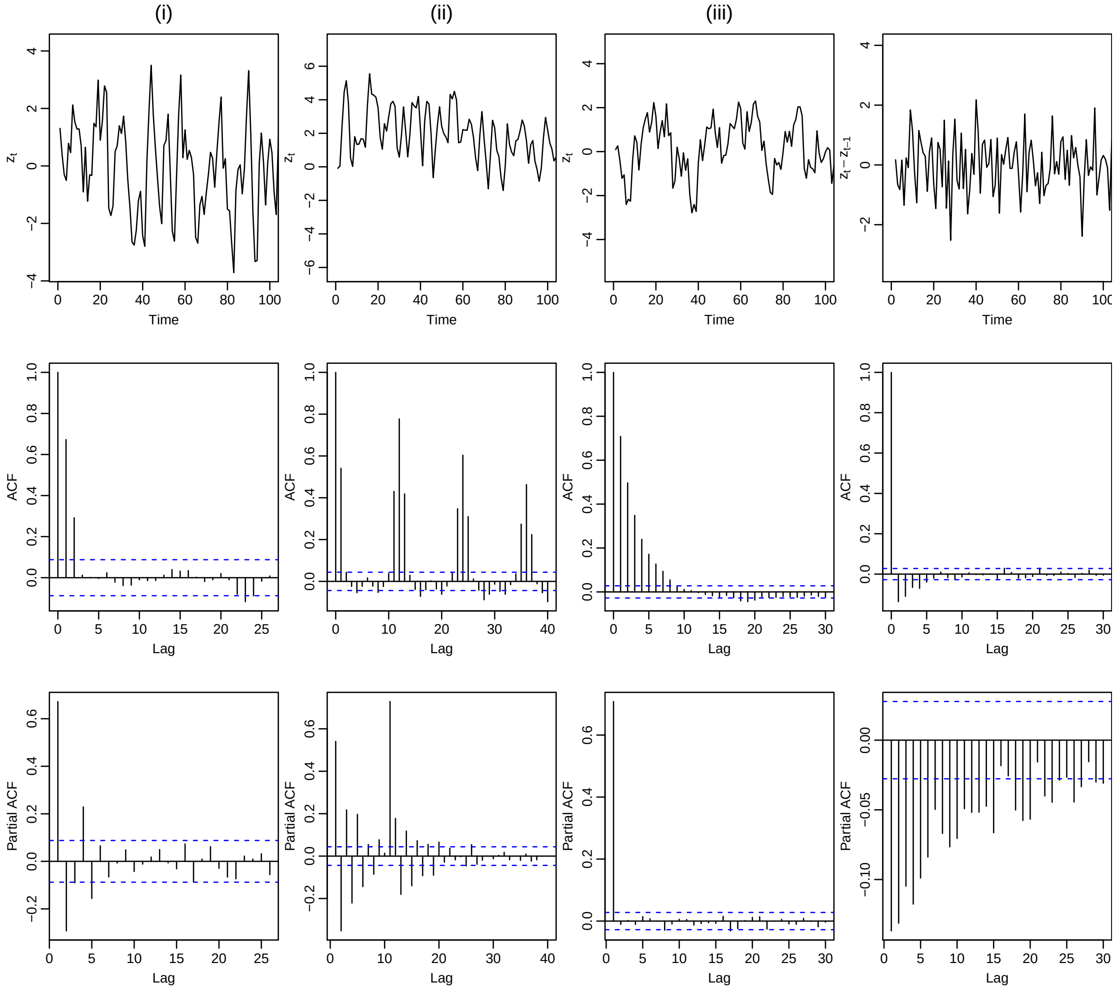

Figure 1: Three different times series (i), (ii) and (iii) (first row) and their associated sample autocorrelation (second row) and sample partial autocorrelation functions (third row). The forth column shows series (iii) after first order differencing.

Problem 1

a) Choose the order of ARIMA(p, d, q) models or seasonal ARIMA$(p, d, q) \times (P, D, Q)_s$ models if you were to fit such models to datasets (i) (first column), (ii) (second colum) and (iii) (third and forth column) in Fig. 1. Dataset (ii) contains monthly observations.

b) Is it necessary to apply differencing to dataset (iii)? Given your choice of model for dataset (iii), what ARIMA(p, d, q) model describes the behaviour of the differenced series in column four? Verify that your conclusion is consistent with the sample ACF and PACF of the differenced series.

Problem 2 Consider the process $\{Z_t\}$ defined by

$$Z_t = (1 - \theta_1 B - \theta_2 B^2)a_t \tag{1}$$

where $\theta_1 = \frac{5}{2}$, $\theta_2 = -1$ and $\{a_t\}$ is zero-mean Gaussian white noise with variance $\sigma_a^2 = 1$. B is the backshift operator.

a) Compute the autocovariance function γ_k of the process.

b) Compute the partial autocorrelations ϕ_{kk} for lags $k \leq 2$.

c) Consider an MA(2)-model for which the MA-polynomial $\theta'(B) = 1 - \theta_1' B - \theta_2' B^2$ has a single root of multiplicity two, $B_1 = B_2 = 2$. Find the necessary parameter values θ_1', θ_2' and $\sigma_a^{2'}$ such that this model has the same autocovariance function as model (1). Which of the two models are preferable?

d) Find a general numerical formula for all coefficients of the AR(∞) representation of the preferred model from point c. Explain how this can be used to compute the one- and two-step ahead forecast of Z_{n+1} and Z_{n+2} given observations $Z_n, Z_{n-1}, \ldots$ of all the past up to the forecast time origin n. Also derive the forecast for lead times $l > 2$. Find the variances of the associated forecast errors.

Problem 3 Consider the state space model with state equation

$$Y_t = AY_{t-1} + a_t \tag{2}$$

and observation equation

$$Z_t = Y_t + b_t \tag{3}$$

where $A = 0.7$, a_t is zero-mean Gaussian white noise with variance $\Sigma = 1$ and b_t is Gaussian white noise with variance $\Omega = 0.5$.

Let $\hat{Y}_{t|s} = E(Y_t|Z_1, Z_2, \ldots, Z_s)$ and $V_{t|s} = \mathrm{Var}(Y_t|Z_1, Z_2, \ldots, Z_s)$. Figure 2 shows 10 observations $Z_1, Z_2, \ldots, Z_{10}$ from the model, and, for $t = 1, 2, \ldots, 9$, $\hat{Y}_{t|t}$ (solid line), $\hat{Y}_{t|t} \pm 1.96\sqrt{V_{t|t}}$ (dashed lines). For $t = 9$, $\hat{Y}_{t|t} = 2$ and $V_{t|t} = 0.4$.

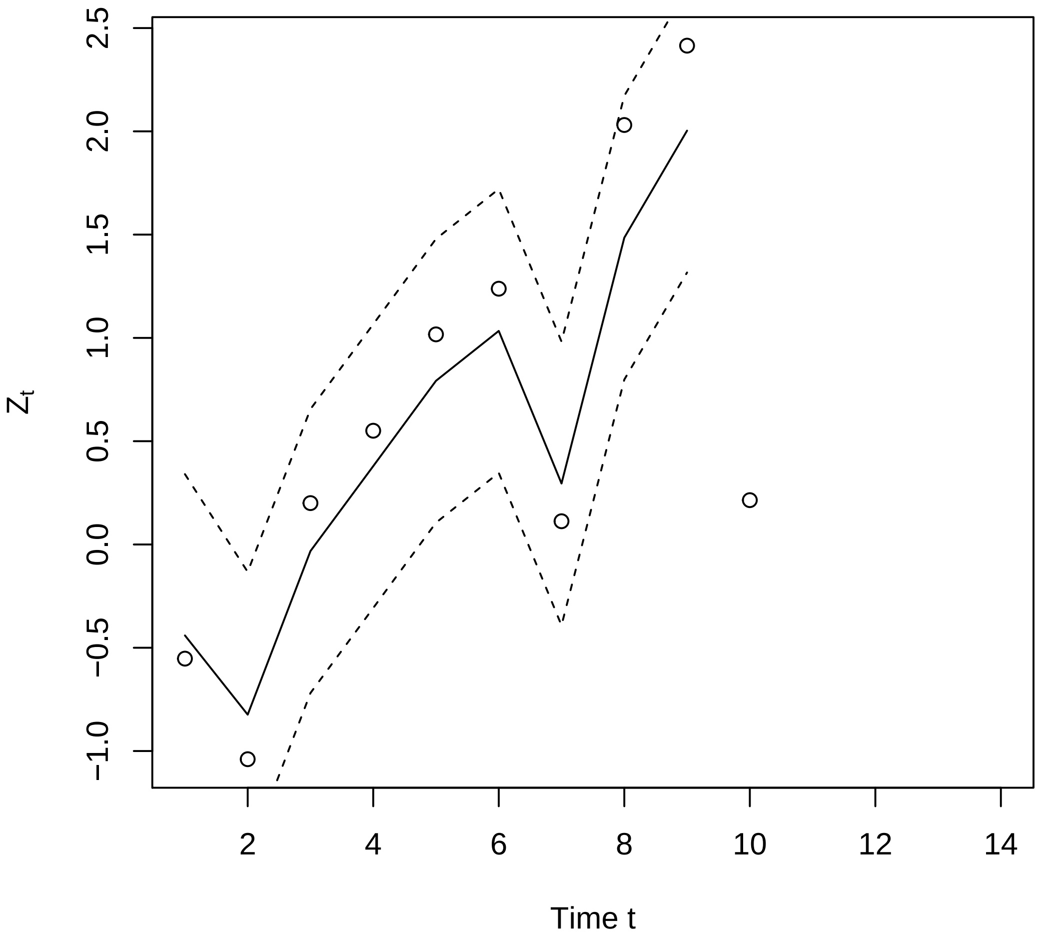

Figure 2: Ten observations (circles) from the state space model in the text and filtered values (solid line) with associated 95%-probability limits (dashed line) up to $t = 9$.

a) Compute a forecast of Y_{10} and the associated variance of the forecast error based on $Z_1, Z_2, \ldots, Z_9$. Also compute the conditional mean and variance of Y_{10} when also conditioning on $Z_{10} = 0.2$. Briefly discuss the behaviour of the computed conditonal mean values.

b) Briefly explain how the total log likelihood of the above state space model can be computed via the Kalman filter. Given the parameter values specifed above, compute the contribution to the log likelihood from observations Z_{10}.

c) Using all the data, compute a forecast of Y_{14} and the variance of the associated forecast error.

Problem 4 Consider the ARCH(1) model,

$$\sigma_t^2 = \mathrm{Var}(\eta_t|\eta_{t-1}) = \theta_0 + \theta_1\eta_{t-1}^2,$$
$$\eta_t = \sigma_t e_t$$

where $\{e_t\}$ is standard normal white noise, $\theta_0 > 0$ and $0 \leq \theta_1 < 1$. Letting $a_t = \eta_t^2 - \sigma_t^2$, this model can also be written in autoregressive form as

$$\eta_t^2 = \theta_0 + \theta_1\eta_{t-1}^2 + a_t. \tag{4}$$

a) Show that the unconditional variance of η_t is given by $\theta_0/(1-\theta_1)$. Also find an expression for the conditonal variance $\mathrm{Var}(a_t|\eta_{t-1})$.

Appendix: The Kalman recursions:

Model:

$$\begin{aligned} \mathbf{Y}_{t+1} &= \mathbf{A}\mathbf{Y}_t + \mathbf{G}\mathbf{a}_{t+1}, \qquad & \mathbf{a}_t &\sim N(\mathbf{0}, \mathbf{\Sigma}) \\ \mathbf{Z}_t &= \mathbf{H}\mathbf{Y}_t + \mathbf{b}_t, \qquad & \mathbf{b}_t &\sim N(0, \mathbf{\Omega}) \end{aligned}$$

Forecasting, $t \geq s$:

$$\begin{aligned} \hat{\mathbf{Y}}_{t+1|s} &= \mathbf{A}\hat{\mathbf{Y}}_{t|s}, \\ \mathbf{V}_{t+1|s} &= \mathbf{A}\mathbf{V}_{t|s}\mathbf{A}^\top + \mathbf{G}\mathbf{\Sigma}\mathbf{G}^\top. \end{aligned}$$

Filtering:

$$\begin{aligned} \hat{\mathbf{Y}}_{t+1|t+1} &= \hat{\mathbf{Y}}_{t+1|t} + \mathbf{K}_{t+1}(\mathbf{Z}_{t+1} - \mathbf{H}\hat{\mathbf{Y}}_{t+1|t}), \\ \mathbf{V}_{t+1|t+1} &= (\mathbf{I} - \mathbf{K}_{t+1}\mathbf{H})\mathbf{V}_{t+1|t}, \\ \mathbf{K}_{t+1} &= \mathbf{V}_{t+1|t}\mathbf{H}^\top(\mathbf{H}\mathbf{V}_{t+1|t}\mathbf{H}^\top + \mathbf{\Omega})^{-1}. \end{aligned}$$

Smoothing:

$$\begin{aligned} \hat{\mathbf{Y}}_{t|n} &= \hat{\mathbf{Y}}_{t|t} + \mathbf{J}_t(\hat{\mathbf{Y}}_{t+1|n} - \hat{\mathbf{Y}}_{t+1|t}), \\ \mathbf{V}_{t|n} &= \mathbf{V}_{t|t} + \mathbf{J}_t(\mathbf{V}_{t+1|n} - \mathbf{V}_{t+1|t})\mathbf{J}_t^\top, \\ \mathbf{J}_t &= \mathbf{V}_{t|t}\mathbf{A}^\top\mathbf{V}_{t+1|t}^{-1}. \end{aligned}$$