Problem 1

a) (i) The acf cuts off for lags $k > 2$ and the PACF tails offs which suggest an ARIMA$(0, 0, 2)$ model

(ii) The acf has peaks at multiples of 12 which suggests a seasonal model with period $s = 12$. At multiples of s the acf tails of where as the pacf has a single peak around $1s$[1] which suggest that the seasonal part of the model is AR(1). Around each peak the acf cuts of after lag $k > 1$ and the pacf tails off which suggest that the regular part of the model is MA(1). Overall, an ARIMA$(0, 0, 1) \times (1, 0, 0)_{12}$ model may be appropriate.

(iii) The acf tails off and the pacf cuts of after lags $k > 1$. Hence, this is a simple AR(1) process.

b) The acf of the non-differenced data does not tail off slowly so no differencing is needed. Given that the the process is AR(1),

$$(1 - \phi_1 B)Z_t = a_t.$$

Applying the difference operator $1 - B$ yields

$$(1 - \phi_1 B)(1 - B)Z_t = (1 - B)a_t$$

and so the differenced series $w_t = (1 - B)Z_t$ satisfies

$$(1 - \phi_1 B)w_t = (1 - B)a_t,$$

that is, an ARMA(1,1) model. This agrees well with the observed sample acf and pacf of w_t which both exhibit approximately geometric tailing off behaviour for lags $k > q = p = 1$.

[1] Counterintuitively, the peak in the pacf is at lag 11. Doing `acf2AR(ARMAacf(ar=c(rep(0,11),.8),ma=c(.9),lag.max=12))` yields $\phi_{11,11} = 0.77$, $\phi_{11,12} = 0.584$ and $\phi_{12,12} = 0.199$.

Problem 2

a) The autocovariance function becomes

$$
\begin{aligned}
\gamma_0 &= \sigma_a^2(1^2 + \theta_1^2 + \theta_2^2) = 1 + \left(\frac{5}{2}\right)^2 + 1 = \frac{33}{4} \\
\gamma_1 &= \sigma_a^2(-\theta_1 + \theta_1\theta_2) = -\frac{5}{2} - \frac{5}{2} = -5 \\
\gamma_2 &= -\sigma_a^2\theta_2 = 1 \\
\gamma_k &= 0 \text{ for } k > 2.
\end{aligned}
$$

b) First computing the autocorrelation function

$$
\begin{aligned}
\rho_0 &= 1 \\
\rho_1 &= \frac{-5 \cdot 4}{33} = -\frac{20}{33} \\
\rho_2 &= \frac{1 \cdot 4}{33} = \frac{4}{33}
\end{aligned}
$$

For lags $k \leq 2$, the partial autocovariance function becomes

$$
\begin{aligned}
\phi_{11} &= \rho_1 = -\frac{20}{33} \\
\phi_{22} &= \frac{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & \rho_2 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix}} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = \frac{\frac{4}{33} - \left(\frac{20}{33}\right)^2}{1 - \left(\frac{20}{33}\right)^2} = -\frac{269}{689} = -0.38
\end{aligned}
$$

c) The MA(2) model

$$
\begin{aligned}
Z_t &= (1 - \frac{1}{2}B)^2 a_t \\
&= (1 - B + \frac{1}{4}B^2)a_t \\
&= (1 - \theta_1' B - \theta_2' B^2)a_t
\end{aligned}
$$

has moving average parameters $\theta_1' = 1$ and $\theta_2' = -\frac{1}{4}$. Hence, the autocovariance function is

$$\begin{aligned}
\gamma_0' &= \sigma_a^{2'}(1^2+\theta_1^{2'}+\theta_2^{2'}) = \sigma_a^{2'}\frac{33}{16} \\
\gamma_1' &= \sigma_a^{2'}(-\theta_1'+\theta_1'\theta_2') = -\sigma_a^{2'}\frac{5}{4} \\
\gamma_2' &= -\sigma_a^{2'}\theta_2' = \sigma_a^{2'}\frac{1}{4} \\
\gamma_k' &= 0 \text{ for } k > 2.
\end{aligned}$$

Choosing $\sigma_a^{2'} = 4$, $\gamma_k' = \gamma_k$ for all k and both models then represent the same stochastic process. While the first model is non-invertible (it has one root inside the unit circle) the second model is invertible as its double root $B = 2$ outside the unit circle. This model is thus preferable in that we can write it in $AR(\infty)$ form.

d) The polynomial in the AR(∞) representation

$$\pi(B)Z_t = a_t$$

must satisfy

$$(1-\pi_1 B - \pi_2 B^2 - \dots)(1 - B + \frac{1}{4}B^2) = 1.$$

Expanding the product and equating coefficients, $\pi_1 = -1$, $\pi_2 = -\frac{3}{4}$ and for $i \geq 3$ the coefficients satisfies the difference equation

$$\theta'(B)\pi_i = 0.$$

With one root of multiplicity two, the general solution is

$$\pi_i = b_1\frac{1}{2^i} + b_2 i\frac{1}{2^i}.$$

Using the initial conditions $\pi_1 = -1$ for $i = 1$ and $\pi_2 = -\frac{3}{4}$ for $i = 2$ we find that $b_1 = b_2 = -1$ and

$$\pi_i = -\frac{1}{2^i} - i\frac{1}{2^i}$$

for $i \geq 1$.

The one-step ahead forecast can be now computed as

$$\begin{aligned}
\hat{Z}_n(1) &= E(Z_{n+1}|Z_n, Z_{n-1}, \dots) \\
&= E(\pi_1 Z_n + \pi_2 Z_{n-1} + \dots | Z_n, Z_{n-1}, \dots) \\
&= \pi_1 Z_n + \pi_2 Z_{n-1} + \dots,
\end{aligned}$$

the two-step ahead forecast as

$$\begin{aligned}\hat{Z}_n(2) &= E(Z_{n+2}|Z_n, Z_{n-1}, \dots) \\ &= E(\pi_1 Z_{n+1} + \pi_2 Z_n + \dots |Z_n, Z_{n-1}, \dots) \\ &= \pi_1 E(Z_{n+1}|Z_n, Z_{n-1}, \dots) + \pi_2 Z_n + \dots \\ &= \pi_1 \hat{Z}_n(1) + \pi_2 Z_n + \dots,\end{aligned}$$

and the l-step ahead forecast as

$$\begin{aligned}\hat{Z}_n(l) &= E(Z_{n+2}|Z_n, Z_{n-1}, \dots) \\ &= E(a_{n+l} - \theta_1 a_{n+l-1} - \theta_2 a_{n+l-2}|Z_n, Z_{n-1}, \dots) \\ &= 0\end{aligned}$$

for $l > q = 2$.

The variance of the forecast error is given by

$$\text{Var}(e_n(l)) = \sigma_a^{2'} \sum_{j=0}^{l-1} \psi_j^2 = \begin{cases} 4 & \text{for } l = 1 \\ 4(1+1) = 8 & \text{for } l = 2 \\ 4(1+1+\frac{1}{4^2}) = \frac{33}{4} & \text{for } l > 2 \end{cases}$$

Problem 3

a) Using the Kalman forecasting recursions, the forecasted mean and variance of Y_{10} becomes

$$\begin{aligned}\hat{Y}_{10|9} &= A\hat{Y}_{9|9} = 0.7 \cdot 2 = 1.4, \\ V_{10|9} &= AV_{9|9}A^T + G\Sigma G^T = 0.7^2 \cdot 0.4 + 1 = 1.2.\end{aligned}$$

As expected from the mean reverting behaviour of the AR(1) state equation, the forecasted mean is closer to zero than the estimated state after the previous filtering step.

The Kalman filtering recursions yields

$$\begin{aligned}K_{10} &= V_{10|9}H^T(HV_{10|9}H^T + \Omega)^{-1} = \frac{1.2}{1.2 + 0.5} = 0.705, \\ \hat{Y}_{10|10} &= \hat{Y}_{10|9} + K_{10}(Z_{10} - H\hat{Y}_{10|9}) = 1.4 + 0.705(0.2 - 1.4) = 0.55, \\ V_{10|10} &= (I - K_{10}H)V_{10|9} = (1 - .705)1.2 = 0.354.\end{aligned}$$

Conditioning also on Z_{10}, the estimated of the state Y_10 is moved towards the observed value of Z_{10} away from the forecasted value.

b) The total likelihood can in general be written as

$$L(\theta) = f(Z_1, Z_2, \dots, Z_n) = f(Z_1) \prod_{t=2}^{n} f(Z_t | Z_{t-1}, \dots, Z_1). \tag{1}$$

For a Gaussian, linear state-space model, the Z_t's are jointly multivariate normal. Hence, the conditional densities in (1) are also Gaussian with means and variances that can be expressed in terms of quantities computed via the Kalman recursions as

$$\begin{aligned} E(Z_t|Z_{t-1}, Z_{t-2}, \dots) &= E(HY_t + b_t|Z_{t-1}, Z_{t-2}, \dots) \\ &= HE(Y_t|Z_{t-1}, Z_{t-2}, \dots) \\ &= H\hat{Y}_{t|t-1} \end{aligned}$$

and

$$\begin{aligned} \operatorname{Var}(Z_t|Z_{t-1}, Z_{t-2}, \dots) &= \operatorname{Var}(HY_t + b_t|Z_{t-1}, Z_{t-2}, \dots) \\ &= H \operatorname{Var}(Y_t|Z_{t-1}, Z_{t-2}, \dots)H^T + \Omega \\ &= HV_{t|t-1}H^T + \Omega. \end{aligned}$$

For the parameter values in the present example $E(Z_{10}|Z_9, \dots) = 1.4$, $\operatorname{Var}(Z_{10}|Z_9, \dots) = 1.2 + 0.5 = 1.7$ such that the contribution the log likelihood becomes

$$-\frac{1}{2}\ln(2\pi 1.7) - \frac{(0.2 - 1.4)^2}{2 \cdot 1.7} = -1.607.$$

c) Using the Kalman forecasting recursions it follows that $\hat{Y}_{14|10} = 0.7^4\hat{Y}_{10|10} = 0.132$. The variance can be computed as

$$\begin{aligned} V_{11|10} &= 0.7^2 V_{10|10} + 1 = 0.7^2 \cdot 0.354 + 1 = 1.173 \\ V_{12|10} &= 0.7^2 V_{11|10} + 1 = 1.575 \\ V_{13|10} &= 0.7^2 V_{12|10} + 1 = 1.772 \\ V_{14|10} &= 0.7^2 V_{13|10} + 1 = 1.868 \end{aligned}$$

Problem 4

a) Using the law of total variance,

$$\begin{aligned}\operatorname{Var}\eta_t &= E\operatorname{Var}(\eta_t|\eta_{t-1}) + \operatorname{Var}E(\eta_t|\eta_{t-1})\\ &= E(\theta_0 + \theta_1\eta_{t-1}^2) + \operatorname{Var}0\\ &= \theta_0 + \theta_1\operatorname{Var}\eta_{t-1}.\end{aligned}$$

Assuming that the process is variance stationary, we can solve for $\operatorname{Var}\eta_t = \operatorname{Var}\eta_{t-1} = \theta_0/(1-\theta_1)$.

Writing the ARCH(1) model in autoregressive form,

$$\begin{aligned}\operatorname{Var}(a_t|\eta_{t-1}) &= \operatorname{Var}(\eta_t^2 - \sigma_t^2|\eta_{t-1})\\ &= \operatorname{Var}(\sigma_t^2(e_t^2-1)|\eta_{t-1})\\ &= \operatorname{Var}((\theta_0+\theta_1\eta_{t-1}^2)e_t^2|\eta_{t-1})\\ &= 2(\theta_0+\theta_1\eta_{t-1}^2)^2\end{aligned}$$

since e_t^2 is chi-square with one degree of freedom.