NTNU
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Department of Mathematical Sciences

Examination paper for **TMA4285 Time Series Models**

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Examination time (from–to): 09:00–13:00

Permitted examination support material: *Tabeller og formler i statistikk*, Tapir Forlag/ Fagbokforlaget, K. Rottmann: *Matematisk formelsamling*, Calculator Casio fx-82ES PLUS, CITIZEN SR-270X, CITIZEN SR-270X College or HP30S, one yellow A5-sheet with your own handwritten notes.

Other information:
Note that you should explain your reasoning behind your answers. You may write in English and/or Norwegian. You may write with a pencil.

Language: English

Number of pages: 3

Number of pages enclosed: 1

Checked by:

Informasjon om trykking av eksamensoppgave
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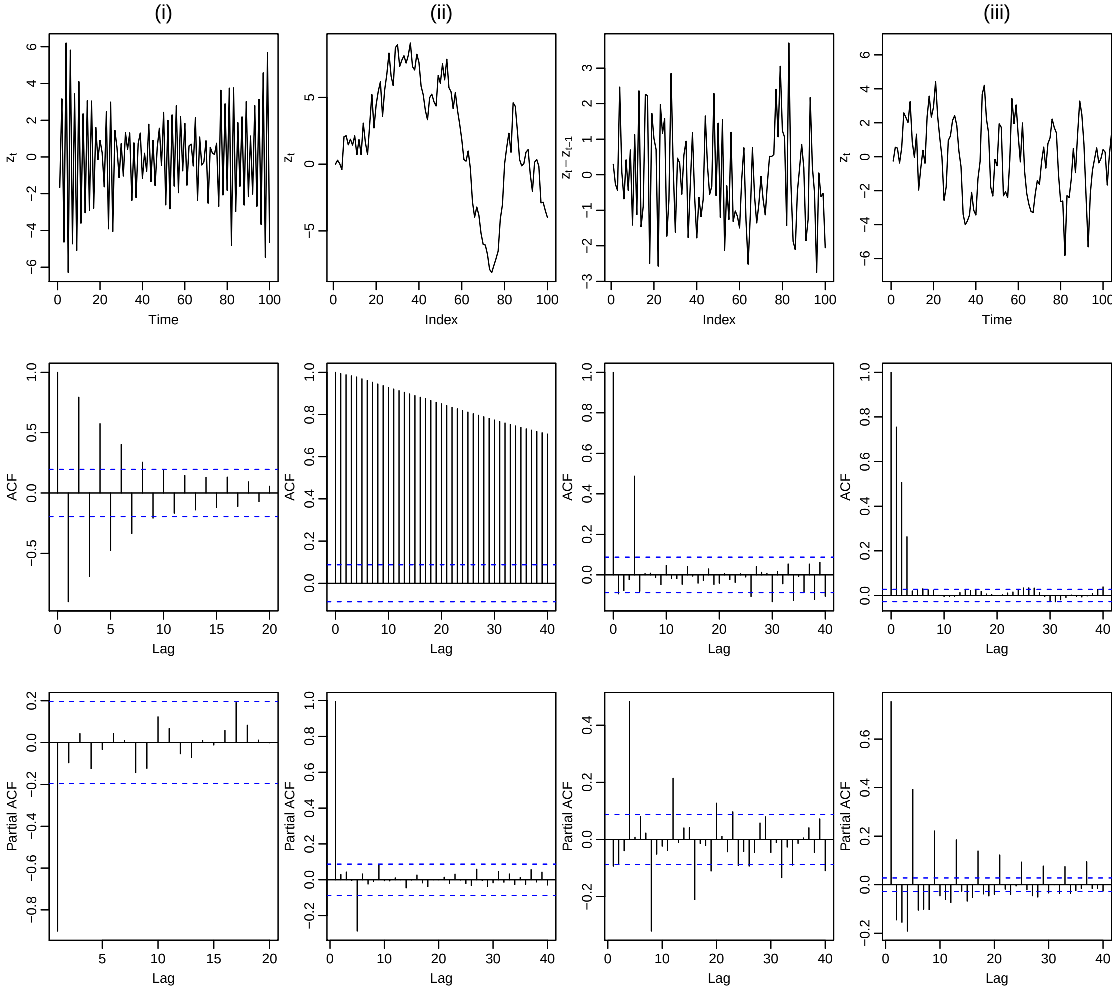

Figure 1: Three different times series (i), (ii) and (iii) (first row) and their associated sample autocorrelation (second row) and sample partial autocorrelation functions (third row). The third column shows series (ii) after first order differencing.

Problem 1

a) Find the autocorrelation function of an MA(3) process defined by

$$Z_t = a_t + a_{t-1} + a_{t-2} + a_{t-3} \tag{1}$$

where a_t is a white noise process.

b) Choose the order of ARIMA(p, d, q) models or seasonal ARIMA$(p, d, q) \times (P, D, Q)_s$ models if you were to fit such models to datasets (i) (first column),

(ii) (second and third column) and (iii) (fourth column) in Fig. 1. Dataset (ii) contains quarterly observations.

c) Based on the observed sample autocorrelation functions in Fig. 1, estimate all model parameters by eye (except σ_a^2) for time series (i), (ii) and (iii).

Problem 2

a) Give a definition of the partial autocorrelation function ϕ_{kk}. Find an expression for ϕ_{33} using Cramer's rule. Use the fact that $\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}$ for $k > 0$ to show that $\phi_{33} = 0$ for an AR(2) process. Hint: Recall that the determinant of a singular matrix is zero.

Problem 3 Suppose that $\{Z_t\}$ follows a process defined by

$$Z_t = Z_{t-1} + a_t - 2a_{t-1} \tag{2}$$

where $\{a_t\}$ is Gaussian white noise with variance $\sigma_a^2 = 0.25$.

a) Is Z_t a stationary process? Why is the model as given by (2) not invertible? Find an alternative represention of the model specified by (2) that is invertible and find the AR(∞)-representation of the invertible model.

Suppose that we have $n = 5$ observations $Z_1, Z_2, \ldots, Z_5$ as shown in Table 1.

t	z_t
1	0.00
2	0.72
3	0.12
4	1.75
5	0.32

Table 1: Observed values of Z_t in problem 3

b) Compute the numerical value of the infinite history 1-step ahead forecast of Z_6 neglecting terms containing $Z_0, Z_{-1}, \ldots$. Find the variance of the associated forecast error. Also derive a general formula for the l-step ahead forecast for $l > 1$.

c) Derive the pure moving average representation of the model

$$Z_t = \sum_{j=0}^{\infty} \psi_j a_{t-j} \tag{3}$$

using methods assuming that the process is stationary. Is (3) a valid representation of (2)?

Derive a general formula for variance of the error of the l-step ahead forecast in point (b).

d) Show that a possible interpretation of model (2) is that $\{Z_t\}$ is a first order moving average $Z_t = \xi_t - \theta_1 \xi_{t-1}$ of a random walk process $\{\xi_t\}$ defined by $(1 - B)\xi_t = a_t$. Use this to construct a state-space representation (see the Appendix) of the above non-invertible model (with $\theta_1 = 2$ and $\sigma_a^2 = 0.25$) with $Y_t = \begin{bmatrix} \xi_t \\ \xi_{t-1} \end{bmatrix}$ as state vector.

e) Suppose that we have run the Kalman forecasting and filtering recursions on the state-space model in point (d) using the data in Table 1 up to time $t = 5$ such that we know that

$$\hat{Y}_{5|5} = \begin{bmatrix} -0.668 \\ -0.494 \end{bmatrix}, \qquad V_{5|5} = \begin{bmatrix} 0.7507 & 0.3754 \\ 0.3754 & 0.1877 \end{bmatrix} \tag{4}$$

in the notation given in the Appendix. Find $\hat{Y}_{6|5}$ and use this to compute a forecast of Z_6, that is, $E(Z_6|Z_1, \ldots, Z_5)$.

Also compute $V_{6|5}$ and use this to find the variance of the error of the forecast of Z_6.

Discuss how these results compare to the forecast in point (b).

f) Discuss how you would chooose the initial values $\hat{Y}_{1|0}$ and $V_{1|0}$ assuming that you only have vague prior knowledge about the state vector at time $t = 1$, $Y_1 = \begin{bmatrix} \xi_1 \\ \xi_0 \end{bmatrix}$. In particular, should ξ_1 and ξ_0 be independent?

Appendix: The Kalman recursions:

Model:

$$\begin{aligned} Y_t &= AY_{t-1} + Ga_t, & a_t &\sim N(0, \Sigma) \\ Z_t &= HY_t + b_t, & b_t &\sim N(0, \Omega) \end{aligned}$$

Notation:

$$\begin{aligned} \hat{Y}_{t|s} &= E(Y_t | Z_1, Z_2, \dots, Z_s) \\ V_{t|s} &= \mathrm{Var}(Y_t | Z_1, Z_2, \dots, Z_s) \end{aligned}$$

Forecasting, $t \geq s$:

$$\begin{aligned} \hat{Y}_{t+1|s} &= A\hat{Y}_{t|s}, \\ V_{t+1|s} &= AV_{t|s}A^\top + G\Sigma G^\top. \end{aligned}$$

Filtering:

$$\begin{aligned} \hat{Y}_{t+1|t+1} &= \hat{Y}_{t+1|t} + K_{t+1}(Z_{t+1} - H\hat{Y}_{t+1|t}), \\ V_{t+1|t+1} &= (I - K_{t+1}H)V_{t+1|t}, \\ K_{t+1} &= V_{t+1|t}H^\top (HV_{t+1|t}H^\top + \Omega)^{-1}. \end{aligned}$$

Smoothing:

$$\begin{aligned} \hat{Y}_{t|n} &= \hat{Y}_{t|t} + J_t(\hat{Y}_{t+1|n} - \hat{Y}_{t+1|t}), \\ V_{t|n} &= V_{t|t} + J_t(V_{t+1|n} - V_{t+1|t})J_t^\top, \\ J_t &= V_{t|t}A^\top V_{t+1|t}^{-1}. \end{aligned}$$