NTNU
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4285 Time Series Models**

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Examination date: November 27, 2019

Examination time (from–to): 15:00 – 19:00

Permitted examination support material: C

— Tabeller og formler i statistikk, Akademika
— K.Rottman. Matematisk formelsamling
— Stamped yellow A5 sheet with your own handwritten notes
— Determined, single calculator

Other information:

You may write in English or Norwegian.

All answers have to be justified. They should include enough details in order to see how they have been obtained.

All 20 sub-problems carry the same weight for grading.

Language: English

Number of pages: 3

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
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Date Signature

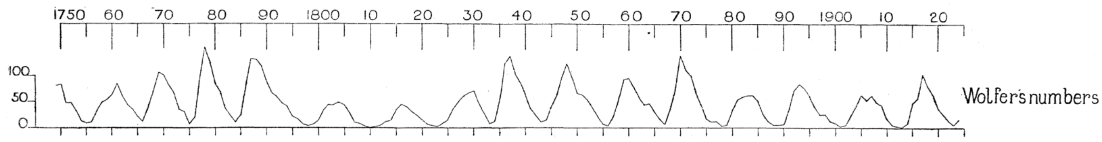

Figure 1: A time series.

Problem 1 Let

$$X_t - 1.318X_{t-1} + 0.634X_{t-2} = Z_t, \quad t \in \mathbb{Z}, \tag{1}$$

with $Z \sim \text{WN}(0, 289.2)$.

a) State the needed additional requirements ensuring that $X \sim \text{ARMA}(p, q)$.

b) Determine p and q.

c) Prove that X is causal.

d) Calculate ψ_j for $j \leq 2$ where $X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$.

e) Determine a finite difference equation for the covariance function γ of X.

f) Calculate and graph the partial correlation function of X given that $\gamma(0) = 1384.1$, $\gamma(1) = 1116.4$, $\gamma(2) = 593.9$.

g) Derive equations that determines the best linear predictor $F(h) = \hat{X}_{n+h}$ of X_{n+h} based on $X_1, X_2, \ldots, X_n$.

h) Calculate the prediction uncertainty u of $\hat{X}_{n+1}$.

Problem 2 Sunspots are regions of reduced surface temperature that appear as dark spots on the Sun. Figure 1 shows the Wolfer's sunspot numbers analyzed by George Udny Yule in 1927. Let $s = (s_1, \ldots, s_{100})$ be the sunspot numbers for the years 1770–1869. It will be assumed that s is a sample from a weakly stationary causal stochastic process S. The data s gives the sample statistics $\hat{\mu} = 46.93$, $\hat{\gamma}(0) = 1382.2$, $\hat{\gamma}(1) = 1114.4$, $\hat{\gamma}(2) = 591.73$, and the sample partial correlation function $\hat{\alpha}$ in Figure 2.

a) How is the sample mean $\hat{\mu}$ calculated?

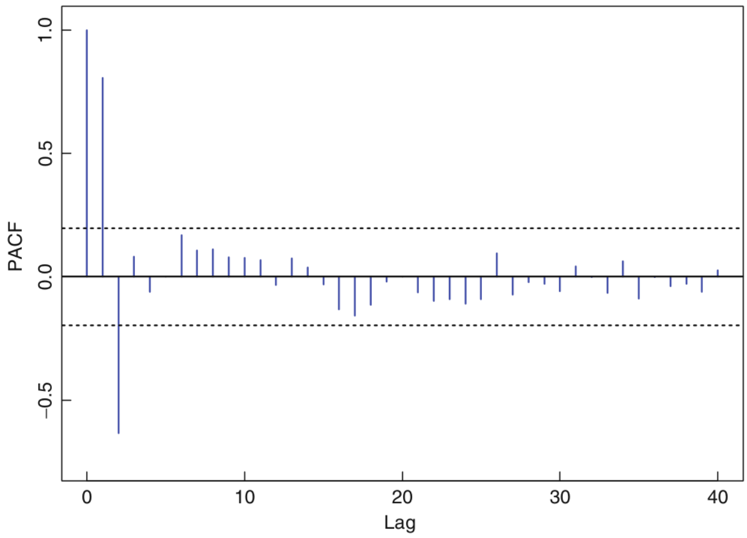

Figure 2: A sample partial correlation function.

b) How is the sample covariance function $\hat{\gamma}$ calculated?

c) Justify that $\hat{\mu}$ is an unbiased estimator for the mean μ.

d) Derive a formula for estimating the uncertainty of $\hat{\mu}$.

e) Figure 2 gives an argument for assuming $X = S - \mu \sim \text{AR}(2)$. Explain this.

f) Estimate the parameters of X using $\hat{\gamma}(0)$, $\hat{\gamma}(1)$, and $\hat{\gamma}(2)$.

g) Calculate $\hat{\alpha}(0)$, $\hat{\alpha}(1)$, and $\hat{\alpha}(2)$. Compare with Figure 2.

h) Derive explicit formulas for obtaining the maximum likelihood estimates of the parameters of S.

Problem 3 Let Z be a causal stationary GARCH(p, q) process with $Z = \sqrt{h}e$, $e \sim \text{IID}(0, 1)$, and

$$h = \alpha_0 + \alpha(B)Z^2 + \beta(B)h \tag{2}$$

Assume furthermore $\alpha_1 + \cdots + \alpha_p + \beta_1 + \cdots + \beta_q < 1$ and $\operatorname{Var} Z^2 < \infty$. It can then be shown that $[Z^2 - \text{E}(Z^2)] \sim \text{ARMA}(m, q)$ with $m = \max(p, q)$ and

$$Z^2 = \alpha_0 + (\alpha + \beta)(B)Z^2 + U - \beta(B)U \tag{3}$$

with $U = Z^2 - h \sim \text{WN}$.

Let $Z \sim \text{WN}(0, \sigma^2)$ be the estimated noise process obtained for the sunspot numbers with corresponding observed values $z_1, \ldots, z_{100}$. The sample correlation function of $z_1^2, \ldots, z_{100}^2$ indicates that Z^2 is not white noise. Further analysis and estimation indicate that $Z^2 \sim \text{ARMA}(1, 1)$. This motivates to consider the GARCH$(1, 1)$ model $Z = \sqrt{h}e$ with $e \sim \text{IID}(0, 1)$, and

$$h_t = 31.152 + 0.223Z_{t-1}^2 + 0.596h_{t-1} \tag{4}$$

a) Is it reasonable to assume that Z is Gaussian?

b) Can Figure 1 be used to motivate modeling with GARCH noise?

c) Explain how the AICC generalizes the maximum likelihood, and can be used for obtaining possibly improved models using equation (3).

d) Discuss briefly possible alternative models for the sunspot numbers.