16/12-19 Exam in Time Series

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Problem 1

a) X must be weakly stationary (WS). Otherwise (1) allows for adding a general solution of the homogeneous equation resulting in a solution which is not weakly stationary. WS: First two moments are shift invariant.

b) $p = 2$ & $q = 0$ since $X \sim AR(2)$.

c) $1 - \varphi_1 \cdot \lambda - \varphi_2 \lambda^2 = 0$

$$\lambda = \frac{\varphi_1 \pm \sqrt{\varphi_1^2 + 4\varphi_2}}{-2\varphi_2}$$

$$= \frac{1.318 \pm \sqrt{1.318^2 - 4 \cdot 0.634}}{2 \cdot 0.634}$$

$$\approx 1.0394 \pm i \cdot 0.7049$$

$|\lambda| \approx 1.2559 > 1$): Causal

Note : $\angle \lambda \approx \tan^{-1} \frac{0.7049}{1.0394} = 34{,}1443°$

$360°/\angle\lambda \approx 10{,}5435$ corresponding well to approximate 11 year quasi-periodicity as seen in Fig. 1

Proof of causality:

$$(1-\lambda_1^{-1}B)(1-\lambda_2^{-1}B)\,X = Z$$

$$X = (1-\lambda_1^{-1}B)^{-1}(1-\lambda_2^{-1}B)^{-1}\,Z$$

$$= \left(\sum_{j\geq 0}(\lambda_1^{-1}B)^j\right)\left(\sum_{k\geq 0}(\lambda_2^{-1}B)^k\right) Z$$

$$= \sum_{j\geq 0} \psi_j B^j Z$$

is valid when $|\lambda_i|^{-1} < 1$ or equivalently $|\lambda_i| > 1$ from the geometric series convergence.

d) $\psi_j = 0$ for $j < 0$ from causality.

$$\varphi(B)\psi(B)\,X = Z$$

gives the difference equation

$$\varphi(B)\psi = \delta_0$$

$$\psi_j = \delta_0(j) + \varphi_1\psi_{j-1} + \varphi_2\psi_{j-2}$$

$$\psi_0 = 1$$

$$\psi_1 = \varphi_1 = 1.318 = 1.318$$

$$\psi_2 = 1.318 \cdot 1.318 - 0.634 = 1.103$$

e) $\gamma(h) = E(X_{t-h} X_t)$

$$X_t - \varphi_1 X_{t-1} - \varphi_2 X_{t-2} = Z_t$$

(*) $\gamma(h) - \varphi_1 \gamma(h-1) - \varphi_2 \gamma(h-2) = E(X_{t-h} Z_t)$

$h=0$: $\gamma(0) - \varphi_1 \gamma(1) - \varphi_2 \gamma(2) = \sigma^2 \quad (\psi_0 = 1)$

$h=1$: $\gamma(1) - \varphi_1 \cdot \gamma(0) - \varphi_2 \gamma(1) = 0$

$h=2$: $\gamma(2) - \varphi_1 \cdot \gamma(1) - \varphi_2 \cdot \gamma(0) = 0$

This is three equations for three unknowns.
$\varphi_1 = 1.318$, $\varphi_2 = -0.634$, $\sigma^2 = 289.2$ give

$\gamma(0) = 1384.1 \qquad \gamma(1) = 1116.4 \qquad \gamma(2) = 593.7$

(*) gives the FD :

$$\varphi(B)\gamma(h) = 0 \qquad h \geq 2$$

and $\gamma(0)$ & $\gamma(1)$ determines the unique solution. They are determined by :

$\sigma^2 = \gamma_0 - \varphi_1 \gamma_1 - \varphi_2 \cdot [\varphi_1 \gamma_1 + \varphi_2 \gamma_0] = [1 - \varphi_2^2]\gamma_0 - \varphi_1[1+\varphi_2]\gamma_1$

$0 = \gamma_1 - \varphi_1 \gamma_0 - \varphi_2 \gamma_1 = -\varphi_1 \gamma_0 + [1 - \varphi_2]\gamma_1$

$$\gamma_1 = \gamma_0 \cdot \frac{\varphi_1}{1-\varphi_2} \quad \& \quad \sigma^2 = \gamma_0 \cdot [1 - \varphi_2^2 - \varphi_1 \cdot (1+\varphi_2) \cdot \frac{\varphi_1}{1-\varphi_2}]$$

$$\gamma_0 = \sigma^2 \cdot \frac{1-\varphi_2}{1+\varphi_2} \cdot [(1-\varphi_2)^2 - \varphi_1^2]^{-1}, \quad \gamma_1 = \sigma^2 \cdot \frac{\varphi_1}{1+\varphi_2} \cdot [\cdot]^{-1}$$

f) $\alpha(h) = \varphi_{hh}$ is determined by the best linear predictor. Note in particular that

$$X_3 = \varphi_1 X_2 + \varphi_2 X_1 + Z_3$$

gives

$$\hat{X}_3 = \varphi_1 X_2 + \varphi_2 X_1$$

so $\alpha(0) = 1$ & $\alpha(2) = \varphi_2 = -0.634$

$\alpha(1)$ is determined by

$$\hat{X}_2 = \alpha(1) X_1$$

Seeing the best linear predictor. This is given by projection:

$$\hat{X}_2 = P_1 X_2 = e_1 \langle e_1, X_2 \rangle = \frac{X_1 \langle X_1, X_2 \rangle}{\gamma(0)}$$

$$= \frac{\gamma(1)}{\gamma(0)} \cdot X_1$$

$$\therefore \alpha(1) = \rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{1116.4}{1384.1} \approx 0.8066$$

The graph is as Figure 2, except that $\alpha(h) = 0$ for $h > 2$.

The Durbin-Levinson algorithm gives an alternative.

g) $n = 1$: $\hat{X}_{1+h} = X_1 \cdot \gamma(h)/\gamma(0)$ from above argument. $= \rho(h) \cdot X_1$

$n \geq 2$: $\hat{X}_{n+h} = \varphi_1 \cdot \hat{X}_{n+h-1} + \varphi_2 \hat{X}_{n+h-2}$

(FD) $\phi(B) F = 0$ $F(h) = \hat{X}_{n+h}$

(IC) $F(1) = \varphi_1 X_n + \varphi_2 X_{n-1}$ $(= \hat{X}_{n+1})$

$F(2) = \varphi_1 \hat{X}_{n+1} + \varphi_2 X_n$

Alternative (IC): $F(0) = X_n$
$F(1) =$ as above

or $F(-1) = X_{n-1}$
$F(0) = X_n$

The general solution is on the form:

$$F(h) = |\lambda|^{-h} \cdot (A \cdot \cos \theta h + B \sin \theta h)$$

where A & B are determined by $F(1)$ and $F(2)$, or equiv. from $\varphi_1, \varphi_2, X_n$ & X_{n-1} in (IC).

(from $\lambda^{-h} = (|\lambda| \cdot e^{i\theta})^{-h}$ and values from 1c))

h) $X_{n+1} = \varphi_1 X_n + \varphi_2 X_{n-1} + Z_{n+1}$

$\Rightarrow \hat{X}_{n+1} - X_{n+1} = -Z_{n+1}$

$\Rightarrow$ Prediction uncertainty $n = \sigma = \sqrt{289.2} \simeq 17$

Problem 2

a) $\hat{\mu} = \bar{S} = (S_1 + \cdots + S_n)/n$, $n = 100$

b) Let $y_i = (S_i - \bar{S}) \cdot [1 \le i \le n]$

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-|h|} y_t \cdot y_{t+|h|}$$

The factor $\frac{1}{n}$ could be replaced by $1/(n-1)$, but it should NOT be replaced by a factor depending on h.
$\hat{\gamma}$ is nonnegative definite with this definition.

c) $E(\bar{S}) = \frac{1}{n}(\mu_1 + \cdots + \mu_n) = \mu$
due to weak stationarity.

d) $E(\bar{S} - \mu)^2 = Var(\bar{X})$

$$= E\left[\frac{1}{n}\sum_i X_i \cdot \frac{1}{n}\sum_j X_j\right]$$

$$= \frac{1}{n^2} \cdot \sum_{i,j=1}^{n} \gamma(i-j)$$

$$u = \sqrt{\sum_{|h|<n} \left(1 - \frac{|h|}{n}\right) \hat{\gamma}(h)} \Big/ \sqrt{n}$$

e) $\alpha(h) = 0$ for $h > 2$ for an AR(2), and Figure 2 gives $\hat{\alpha}(h) \approx 0$ for $h > 2$.

f) From 1e):

$$h = 0: \quad \gamma(0) - \varphi_1 \gamma(1) - \varphi_2 \gamma(2) = \sigma^2 \quad (\psi_0 = 1)$$
$$h = 1: \quad \gamma(1) - \varphi_1 \cdot \gamma(0) - \varphi_2 \gamma(1) = 0 \quad \} \cdot \gamma(0)$$
$$h = 2: \quad \gamma(2) - \varphi_1 \cdot \gamma(1) - \varphi_2 \cdot \gamma(0) = 0 \quad \} - \cdot \gamma(1)$$

This is the Yule-Walker equations for φ & σ when γ is given. The last two equations are linear and determine φ_1 & φ_2:

$$\gamma_0 \gamma_1 - \varphi_1 \cdot \gamma_0^2 - \gamma_1 \gamma_2 + \varphi_1 \gamma_1^2 = 0$$

$$\varphi_1 = [\gamma_1 \gamma_2 - \gamma_0 \gamma_1] / [\gamma_1^2 - \gamma_0^2]$$

$$\hat{\varphi}_1 \approx [1114,4 \cdot 591,73 - 1382,2 \cdot 1114,4] / [1114,4^2 - 1382,2^2]$$
$$\approx 1,31755 \approx 1,318$$

$$\hat{\varphi}_2 = [\hat{\gamma}_2 - \hat{\varphi}_1 \hat{\gamma}_1] / \hat{\gamma}_0 = [591,73 - \hat{\varphi}_1 \cdot 1114,4] / 1382,2$$
$$\approx -0,63417 \approx -0,634$$

$$\hat{\sigma}^2 = \hat{\gamma}_0 - \hat{\varphi}_1 \hat{\gamma}_1 - \hat{\varphi}_2 \hat{\gamma}_2 = 1382,2 - \hat{\varphi}_1 \cdot 1114,4 - \hat{\varphi}_2 \cdot 591,73 \approx 289,1791$$
$$\approx 289,18$$

The coefficients φ_1 and φ_2 may alternatively be found from the Durbin-Levinson algorithm since $\hat{X}_3 = \varphi_1 X_2 + \varphi_2 X_1$ is the best linear predictor of X_3. The mean square error gives σ^2. The innovations algorithm or direct linear system are more tedious alternatives.

g) Similar to 1 f):

$$\hat{\alpha}(0) = \underline{\underline{1}}, \quad \hat{\alpha}(1) = \hat{\gamma}(1)/\hat{\gamma}(0) \approx \underline{\underline{0{,}8063}}$$

$$\hat{\alpha}(2) = \hat{\varphi}_2 \approx \underline{\underline{-0{,}634}}$$ which

corresponds well with Figure 2.

h) $$L = \prod_{i=1}^{n} f_i(x_i \mid x_{i-1}, \dots, x_1) \qquad (1)$$

In the Gaussian AR(2) case (using 1e):

$$f_i = \frac{1}{\sqrt{2\pi}\,\sigma_i} e^{-\frac{(x_i - \hat{x}_i)^2}{2\sigma_i^2}}; \qquad (2)$$

$$\hat{x}_i = \varphi_1 x_{i-1} + \varphi_2 x_{i-2}, \quad \sigma_i = \sigma \quad i = 3, \dots, n \qquad (3)$$

$$\hat{x}_2 = \alpha_1 x_1, \quad \sigma_2^2 = \sigma_1^2 \cdot [1 - \alpha_1^2], \quad \alpha_1 = \frac{\varphi_1}{1 - \varphi_2}$$

$$\hat{x}_1 = 0, \quad \sigma_1^2 = \gamma_0 = \sigma^2 \cdot \frac{1 - \varphi_2}{1 + \varphi_2} \cdot [(1 - \varphi_2)^2 - \varphi_1^2]^{-1}$$

With $x_i = s_i - \mu$ this gives $L = L(\mu, \varphi_1, \varphi_2, \sigma)$ and this determines the ML estimates.

Problem 3

a) No. If Gaussian, then Z^2 should also be white noise.
This is in conflict with $Z^2 \sim ARMA(1,1)$.

b) Figure 1 can be seen as the result of volatility clustering typical for GARCH models. Some time-spans have large variance, and some not.

c) The AICC is derived based on minimization of the Kullback-Leibler distance between two distributions.
The result is the minimalization of

$$AICC = 2k \cdot C_k - 2 \ln L \qquad (*)$$

where k is the number of parameters in the likelihood L and $C_k > 1$ is a correction factor that improves the asymptotic approximation used in the derivation.
For fixed numbers of parameters it follows that $(*)$ gives a justification for using the maxim likelihood.
Additionally it gives an estimation

method for the orders of an ARMA model directly. This can hence be used on the Z^2 process to, using (3), estimate $(\alpha+\beta)$ and β, and hence the orders of alternative GARCH(p,q) models for Z.

d) The previous indicates that the sunspot numbers can be modelled as an ARMA with GARCH noise. The orders of both can be estimated using AICC separately, or alternatively for the full model directly. Figure 1, with all positive values, could also motivate an initial log transformation - or possibly a more general Box-Cox transformation.