25/11-20 TMA4285 Time Series

1 a

$(1+aB)\cdot(1+bB^7) = 1 + aB + bB^7 + abB^8$
$= \theta(B)$ grad 8

$\phi(B) = 1$ grad 0

$\phi(B)X = \theta(B)Z \Rightarrow X \sim ARMA(0,8)$
$= MA(8)$

fordi X også er svakt stasjonært ved

$X_t = Z_t + aZ_{t-1} + bZ_{t-7} + abZ_{t-8}$ (*)

$E(X_t) = 0$ (uavh. av t)

$E(X_{t+h}X_t) = \gamma(h)$ (uavh. av t)
fra regning i 1d.

1b Fra (*) er $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$

d.v.s. avhengig av $\underbrace{\text{forhistorien}}$

$\ldots Z_{-1}, \underbrace{Z_0, \ldots, Z_t}$.

$//$

(1c) $Z = (1+aB)^{-1}(1+bB^7)^{-1}X$

unikt via geometrisk rekke fordi $|a|, |b| < 1$ og det er antatt at $(Z \sim WN(0,\sigma^2))$ Z er svakt stasjonær.

Uten antagelse om svak stasjonaritet er Z ikke unikt gitt. Det finnes 8 lineært uavhengige løsninger til $(1+aB)(1+bB^7)Z = 0$.

Overstående gir $Z_t = \sum_{j\geq 0} \pi_j X_{t-j}$ med $\sum_{j\geq 0} |\pi_j| < \infty$.

$$1 = (1 + az + bz^7 + abz^8)\cdot(1 + \pi_1 z + \pi_2 z^2 + \cdots)$$

$0 = \pi_1 + a \Rightarrow \pi_1 = -a$

$0 = \pi_2 + a\pi_1 + b \Rightarrow \pi_2 = -a\pi_1 - b$

$0 = \pi_3 + a\pi_2 \Rightarrow \pi_3 = -a\pi_2$

$0 = \pi_4 + a\pi_3 \Rightarrow \pi_4 = -a\pi_3$ etc

1d $X_t = Z_t + aZ_{t-1} + bZ_{t-7} + abZ_{t-8}$

$\gamma(-h) = \gamma(h) = \gamma_h = 0$ bortsett fra:

$\gamma_0 = E(X_t X_t) = [1 + a^2 + b^2 + a^2b^2]\sigma^2$

$\gamma_1 = E(X_{t+1} \cdot X_t) = E[(Z_{t+1} + aZ_t + bZ_{t-6} + abZ_{t-7}) \cdot X_t]$

$= (a + ab^2)\sigma^2$

$\gamma_2 = E\, X_t \cdot (Z_{t+2} + aZ_{t+1} + bZ_{t-5} + abZ_{t-6}) = 0$

$\gamma_3 = E X_t \cdot (Z_{t+3} + aZ_{t+2} + bZ_{t-4} + abZ_{t-5}) = 0 = \gamma_4 = \gamma_5$

$\gamma_6 = ab\sigma^2$, $\gamma_7 = (1+a^2)b\sigma^2$, $\gamma_8 = ab\sigma^2$

h	0	1	7	8
γ	0,65	-0,27	-0,26	0.11
ρ	1	-0.41	-0.41	0.16

$$\gamma(h) = \sigma^2 \cdot \begin{cases} 1 + a^2 + b^2 + a^2b^2 & h = 0 \\ a \cdot (1 + b^2) & |h| = 1 \\ ab & |h| = 6 \\ b \cdot (1 + a^2) & |h| = 7 \\ ab & |h| = 8 \\ 0 & \text{ellers} \end{cases}$$

Se egsi nlikhet i Fig. 2 :)

Lag 1 & 7 dominerer.

(1e) $\hat{X}_n$ is the projection of X_n onto the subspace spanned by $1, X_1, \ldots, X_{256}$. It is unique.

[$\hat{X}_n = \sum_{j=0}^{256} c_j X_j \qquad X_0 = 1$

c_j's are not unique except if X_j are lin. independent]

Note: Completness of $\mathcal{H} = L^2(\Omega)$ is not needed in the argument: A finite dimensional subspace is complete.

1g $(1-B)(1-B^7)$ has (8) roots on the unit circle so T cannot be w. stationary.

(1g) No X is, but 8 linear independent solutions of $(1-D)(1-D^7)\,T_H = 0$ gives non-uniqueness. The general solution is $T = T_p + T_H$ where T_p is a particular solution.

1h $\hat{t}_{200} = t_{200}$ with $0 = \sigma_{200}$.

$(1-B)\cdot(1-B^7) = 1 - B - B^7 + B^8$ & (2)

$$\Rightarrow \hat{T}_n = \hat{T}_{n-1} + \hat{T}_{n-7} - \hat{T}_{n-8} + \hat{X}_n \quad (R)$$

where ^ denotes projection on linear span of $1, T_{-7}, T_{-6}, \dots, T_{256}$.
This is the same as the linear span of $1, T_{-7}, \dots, T_0, X_1, \dots, X_{256}$ due to (2).

Basic assumption: $T_{-7}, \dots, T_0$ uncorrelated with X_i. This means that $\hat{X}_n$ can be found as in 1e.

$$\hat{t}_{257} = t_{256} + t_{250} - t_{249} + \hat{x}_{257}$$

and $\sigma^2_{257} = E\left(T_{257} - \hat{T}_{257}\right)^2 = E\left(X_{257} - \hat{X}_{257}\right)^2$

These two can be calculated from the covariance function γ and the innovations algorithm. (Or Durbin-Levinson)

$\hat{t}_{200}$ can be calculated using (R) repeatedly.

The innovations algorithm gives σ^2_{300}.

1 i The key is uncorrelated initial condition: $X \perp T_{-7}, \dots, T_0$

This ensures that $\hat{\epsilon}_n$ is given recursively by previous $\hat{\epsilon}_k$'s and the $\hat{X}_n$ prediction.

2a) The outcome is uncertain and takes integer values for each day.

2b) 0, 1, 2, ..., (but < 8 mill)

2c) Yes, possibly, since simplest model with 0, 1, 2, ... values.
Every second a certain probability of a new detected case.
Conditionally the above is reasonable, but unconditionally probably not.

A reasonable tentative model is hence $Y_{300} \mid Y_s, s<300 \sim \text{Poisson}(\lambda_{300})$

(2d) $P(Y_{t_1} = y_{t_1}, \cdots, Y_{t_k} = y_{t_k})$

$$= P((Y_{t_1} = y_{t_1}) \cap \cdots \cap (Y_{t_k} = y_{t_k}))$$

$$(Y_{t_1} = y_{t_1}) = \{\omega \mid Y_{t_1}(\omega) = y_{t_1}\}$$

defines both.

Note: $(Y_t = y_t)$ is an event since Y_t is a random variable:

$$(Y_t = y_t) = \bigcap_{n=1}^{\infty} \left[[Y_t \leq y_t - \tfrac{1}{n}]^c \cap (Y_t \leq y_t) \right]$$

Hence all probabilities above are defined.

2e Let $B = (Y_{200} = y)$ and

$A = (Y_{-7} = y_{-7}, \ldots, Y_{256} = y_{256})$,

so

$$g(y) = P(B \mid A) = \frac{P(B \wedge A)}{P(A)}$$

2f Median and mean minimize

$$\tilde{\sigma} = \sum_y |\tilde{\mu} - y| \, g(y)$$

and

$$\sigma = \sqrt{\sum_y (\mu - y)^2 g(y)}$$

respectively and can be used to quantify the uncertainty.

$$\mu = \sum y \cdot g(y)$$

, but $\tilde{\mu}$ by optimization

Uncertainty: 2.5% & 97.5% percentiles are also a natural choice - ISO GUM

2g Must prove that

$U = \log(\max(V, 0.1))$

is a random variable if V is.

In our case V takes values $0, 1, 2, \ldots$. This means that U takes values $\log(0.1), \log(1), \log(2)$ ... on $(V=0), (V=1), \ldots$

(so with probabilities $P(V=0), P(V=1), \ldots$.)

i.e.: $$U = \sum_{i=0}^{\infty} u_i \cdot (V=i)$$

$$(U \leq u) = \bigcup_{\substack{i \\ u_i \leq u}} (V=i)$$

which is an event.

Note: There is 1-1 correspondence between U and V. Both are discrete.

3a $t_i = \log(\max(y_i, 0.1))$

$$(1-B)(1-B^7) = 1 - B - B^7 + B^8$$

$$X_i = t_i - t_{i-1} - t_{i-7} + t_{i-8}$$

gir $X_1, \dots, X_{256}$. $n = 256$

To varianter for $\hat{\rho}(h) = \dfrac{\hat{\gamma}(h)}{\hat{\gamma}(0)}$

avhengig av om $E(X_i) = 0$ antas eller ikke. Dette antar her (hvis ikke: $X_i \to X_i - \bar{X}$).

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{i=1}^{n} X_{i+h} \cdot X_i \qquad X_i = 0, \; i > 256$$

(3c) A close look at fig. 1 gives a reason to expect a 7 day season, via approx. 4 periods in each month. See e.g. between Sep & Oct.. The source of the data also makes a 7 day season reasonable. This motivates the D_7 term. The correlation is essentially 0 for $h > 8$, and motivates MA(8), and the simplest combination is then as given.

See also 1d which gives dominating negative peaks at lag 1 & 7.

3c Using, or re-deriving, the formula for $\gamma(0)$, $\gamma(1)$ and $\gamma(7)$ gives three equations for three unknowns.

Algebraic manipulation gives second order equations, and assuming $|a|, |b| < 1$, give:

$$b = -0.3379$$

$$a = -0.6512$$

$$\sigma^2 = 0.3940$$

3d The covariance function is given as function of a, b, σ.
The innov. alg. can be used to calculate

$$L = \prod_{i=1}^{256} f(x_i \mid x_{i-1}, \dots, x_1)$$

since the conditionals are Gaussian if Z is assumed Gaussian.
Minimization gives $\hat{a}, \hat{b}, \hat{\sigma}$,

3e A forecast for T_{300} can be computed as in 1b.
A reasonable forecast is then
$$Y_{300} = \text{floor}(\exp(T_{300}))$$

3f)

$$X_t = Z_t + aZ_{t-1} + bZ_{t-7} + abZ_{t-8}$$

La $Z_{-7}, Z_{-6}, \ldots, Z_0$ velges 'på passelig vis'. Da er

$$Z_1 = X_1 - aZ_0 - bZ_{-6} - abZ_{-7}$$

$$\vdots$$

$$Z_{256} = X_{256} - aZ_{256-1} - bZ_{256-7} - abZ_{256-8}$$

'Passelig vis' kan være i.i.d. fra $N(0, \sigma^2)$.

Et mer standard alternativ er

$$\hat{Z}_i = (X_i - \hat{X}_i)/\tilde{\sigma}_i \quad \text{(residual)}$$

$$= \hat{W}_i$$

5.3.1 BD.

Et tredje: $Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$

og $\hat{Z}_t$ ved å kun bruke de observerte.

3g No, because of Fig. 2

3h z^2 modelled as an ARMA, so yes.

z^2 is correlated in GARCH

3i Initial estimates for z from 3f. A GARCH(1,1) is in the form:

$$z_t = \sqrt{h_t}\, e_t \quad , \; e_t \; IID(0,1)$$

$$(1 - \beta_1 B)\, h_t = \alpha_0 + \alpha_1 z_{t-1}^2 \quad (*)$$

$\alpha_0, \alpha_1, \beta_1$ can be found using the conditional Σ of z. This is determined from

The conditional variance h_t:

$$\tilde{L} = \prod_{t=2}^{n} \frac{1}{\sqrt{h_t}} \varphi\left(\frac{z_t}{\sqrt{h_t}}\right)$$

and (*) determine h_t by recursion.