LØSNING ØVING 3 - 2000

Oppgave 4: $X_1, \ldots, X_n$ uif Bernoulli (p)
(uavhengige, identisk fordelte)

dvs. $P(X_i = 1) = p$, $P(X_i = 0) = 1-p \Rightarrow P(X_i = x \mid p) = p^x(1-p)^{1-x}$
for $x = 0, 1$.

a)

$$\frac{\sqrt{n}\,(\bar{X}-p)}{\sqrt{p(1-p)}} \xrightarrow{D} N(0,1),$$

$$E(X) = p,\ Var(X) = \underset{\substack{\uparrow \\ E(X^2)}}{p} - \underset{\substack{\uparrow \\ EX^2}}{p^2} = p(1-p)$$

CLT (eller SGT på norsk) kan brukes da $\mu = p$ eksisterer og $0 < \sigma^2 < \infty$ for $p \in \langle 0,1 \rangle$. Dvs

$$\frac{\sqrt{n}\,(\bar{X}-\mu)}{\sigma} = \frac{\sqrt{n}\,(\bar{X}-p)}{\sqrt{p(1-p)}} \xrightarrow{D} N(0,1)$$

b)

$$\frac{\sqrt{n}\,(\bar{X}-p)}{\sqrt{\bar{X}(1-\bar{X})}} = \sqrt{\frac{p(1-p)}{\bar{X}(1-\bar{X})}}\ \frac{\sqrt{n}\,(\bar{X}-p)}{\sqrt{p(1-p)}}$$

ønsker å bruke Cramer: $Y_n \xrightarrow{P} a$, $Z_n \xrightarrow{D} Z$
$\Rightarrow Y_n Z_n \xrightarrow{D} aZ$. Vi vet at

$$Z_n = \frac{\sqrt{n}\,(\bar{X}-p)}{\sqrt{p(1-p)}} \xrightarrow{D} N(0,1) \text{ pga SGT i a.}$$

$Y_n = \sqrt{\frac{p(1-p)}{\bar{X}(1-\bar{X})}}$. Fra Khinchin (store talls svake lov)

har vi at $\bar{X} \xrightarrow{P} \mu$, dvs $\bar{X} \xrightarrow{P} p$. Vi kan bruke Slutsky (fra 5.15 forrige øv.): $U_n \xrightarrow{P} a$ og $g(\cdot)$ er kontinuerlig i a, $\Rightarrow g(U_n) \xrightarrow{P} g(a)$.

Dermed $\bar{X} \xrightarrow{P} p \Rightarrow g(\bar{X}) = \sqrt{\frac{p(1-p)}{\bar{X}(1-\bar{X})}} \xrightarrow{P} \sqrt{\frac{p(1-p)}{p(1-p)}} = 1$

$$\underbrace{\sqrt{\frac{p(1-p)}{\bar{X}(1-\bar{X})}}}_{\substack{(\text{Slutsky})\ \downarrow P \\ 1}}\ \underbrace{\frac{\sqrt{n}\,(\bar{X}-p)}{\sqrt{p(1-p)}}}_{\substack{(\text{SGT})\ \downarrow D \\ N(0,1)}} \xrightarrow[\text{(Cramér)}]{D} N(0,1)$$

c) Se på $(\arcsin\sqrt{\ })$ som variansstabiliserende transform.

Vi må bruke teoremet som sier: $\sqrt{n}\,(\bar{X}_n - \mu) \xrightarrow{D} Y$, g deriverbar i $x = \mu$ med $g'(\mu) \neq 0$. Da vil

$$\sqrt{n}\,(g(\bar{X}_n) - g(\mu)) \xrightarrow{D} g'(\mu)\cdot Y. \quad \text{(referert på forelesn.)}$$

Dvs, SGT sier $\frac{\sqrt{n}\,(\bar{X}_n-p)}{\sqrt{p(1-p)}} \xrightarrow{D} U$ der $U \sim N(0,1)$

men da har vi også at

$$\sqrt{n}\,(\bar{X}_n - p) \xrightarrow{D} \sqrt{p(1-p)}\cdot U, \text{ dvs } N(0, p(1-p))$$

fra Cramér($\underbrace{\frac{\sqrt{n}\,(\bar{X}_n-p)}{\sqrt{p(1-p)}}}_{\xrightarrow{D} U} \cdot \underbrace{\sqrt{p(1-p)}}_{\xrightarrow{P} \sqrt{p(1-p)}} \xrightarrow{D} \sqrt{p(1-p)}\cdot U$, der $U \sim N(0,1)$. $Var(\sqrt{p(1-p)}\,U) = p(1-p)$

Videre lar vi $g(x) = 2\arcsin\sqrt{x}$, $g'(x) = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{\sqrt{x}}$

$$\sqrt{n}\,(2\arcsin\sqrt{\bar{X}_n} - 2\arcsin\sqrt{p}) \xrightarrow{D} \frac{1}{\sqrt{p(1-p)}}\cdot\sqrt{p(1-p)}\cdot U$$

Dermed

$$\sqrt{n}\,(2\arcsin\sqrt{\bar{X}_n} - 2\arcsin\sqrt{p}) \xrightarrow{D} U, \quad U \sim N(0,1)$$

d) konfidensintervall

a:

$$P\left(-u_{\frac{\alpha}{2}} < \frac{\sqrt{n}(\bar{X}-p)}{\sqrt{p(1-p)}} < u_{\frac{\alpha}{2}}\right) \approx 1-\alpha$$

$$P\left(\frac{n(\bar{X}-p)^2}{p(1-p)} < u_{\frac{\alpha}{2}}^2\right) \approx 1-\alpha$$

$$P\left((n+u_{\frac{\alpha}{2}}^2)p^2 - (2n\bar{X}+u_{\frac{\alpha}{2}}^2)p + n\bar{X}^2 < 0\right) \approx 1-\alpha$$

nullpkt:

$$p = \frac{2n\bar{X}+u_{\frac{\alpha}{2}}^2 \pm \sqrt{(2n\bar{X}+u_{\frac{\alpha}{2}}^2)^2 - 4(n+u_{\frac{\alpha}{2}}^2)n\bar{X}^2}}{2(n+u_{\frac{\alpha}{2}}^2)}$$

$$= \frac{n\bar{X}+\frac{1}{2}u_{\frac{\alpha}{2}}^2 \pm u_{\frac{\alpha}{2}}\sqrt{\frac{1}{4}u_{\frac{\alpha}{2}}^2 + n\bar{X}(1-\bar{X})}}{n+u_{\frac{\alpha}{2}}^2}$$

$p \in [\text{nedre}, \text{øvre}]$ er $\approx 1-\alpha$ konfidensintervall.

b:

$$P\left(-u_{\frac{\alpha}{2}} < \frac{\sqrt{n}(\bar{X}-p)}{\sqrt{\bar{X}(1-\bar{X})}} < u_{\frac{\alpha}{2}}\right) \approx 1-\alpha$$

$$p \in \left[\bar{X} \pm \sqrt{\frac{\bar{X}(1-\bar{X})}{n}} \cdot u_{\frac{\alpha}{2}}\right]$$

c:

$$P\left(-u_{\frac{\alpha}{2}} < 2\arcsin\sqrt{\bar{X}} - 2\arcsin\sqrt{p} < u_{\frac{\alpha}{2}}\right) \approx 1-\alpha$$

$$P\left(\arcsin\sqrt{\bar{X}} - \tfrac{1}{2}u_{\frac{\alpha}{2}} < \arcsin\sqrt{p} < \arcsin\sqrt{\bar{X}} + \tfrac{1}{2}u_{\frac{\alpha}{2}}\right) \approx 1-\alpha$$

$\arcsin\sqrt{\bar{X}} \in [0, \frac{\pi}{2}]$. Hvis $(\arcsin\sqrt{\bar{X}} \pm \frac{1}{2}u_{\frac{\alpha}{2}}) \in [0, \frac{\pi}{2}]$ går det greit å løse ut vha sin og kvadrer:

$$p \in \left[\sin^2\left(\arcsin\sqrt{\bar{X}} \pm \tfrac{1}{2}u_{\frac{\alpha}{2}}\right)\right]$$

hvis $\arcsin\sqrt{\bar{X}} - \frac{1}{2}u_{\frac{\alpha}{2}} < 0$ må nedre grense bli 0 og
hvis $\arcsin\sqrt{\bar{X}} + \frac{1}{2}u_{\frac{\alpha}{2}} > \frac{\pi}{2}$ må øvre grense bli 1.

NB, dette er ikke hovedsaken her, men vi bør huske på at nedre grense ≥ 0 og øvre grense ≤ 1.

Oppgave 5: $U \sim N(0,1)$; $V \sim \chi^2_p$; U, V uavh.

løsn. s 226 i boka!

a) simultantetthet U, V.

$$f_{U,V}(u,v) = f_U(u) \cdot f_V(v)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}u^2} \cdot \frac{1}{\Gamma(\frac{p}{2})2^{p/2}} v^{\frac{p}{2}-1} e^{-\frac{v}{2}}$$

$$= \frac{1}{\sqrt{2\pi}\,\Gamma(\frac{p}{2})\,2^{p/2}} v^{p/2-1} \cdot e^{-\frac{1}{2}(u^2+v)}$$

b)

$$T = \frac{U}{\sqrt{\frac{V}{p}}} \sim t_p \qquad (V \sim \chi^2_p \text{, dvs } v \geq 0)$$

Skal bruke transformasjonsformelen med T og $W=V$:

$$T = \frac{U}{\sqrt{\frac{V}{p}}}, \quad W = V \qquad (w \geq 0)$$

$$\underbrace{U = T\sqrt{\frac{W}{p}}}_{h_1(T,W)}, \quad \underbrace{V = W}_{h_2(T,W)}$$

$$J = \begin{vmatrix} \frac{\partial h_1}{\partial t} & \frac{\partial h_1}{\partial w} \\ \frac{\partial h_2}{\partial t} & \frac{\partial h_2}{\partial w} \end{vmatrix} = \begin{vmatrix} \sqrt{\frac{W}{p}} & \frac{T}{\sqrt{p}} \cdot \frac{1}{2\sqrt{W}} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{W}{p}}$$

$$f_{T,W}(t,w) = \frac{1}{\sqrt{2\pi}\,\Gamma(\frac{p}{2})\,2^{p/2}} w^{p/2-1} \cdot e^{-\frac{1}{2}(\frac{1}{p}t^2w+w)} \sqrt{\frac{w}{p}}$$

$$= \frac{1}{\sqrt{2\pi}\,\Gamma(\frac{p}{2})\,2^{\frac{p}{2}}\sqrt{p}} w^{\frac{p}{2}-\frac{1}{2}} e^{-\frac{1}{2}w(\frac{1}{p}t^2+1)}$$