Norwegian University of Science and Technology
Department of Mathematical Sciences

English

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TMA4295 Statistical inference

Friday 3 December 2010 9:00–13:00

Permitted aids: Yellow A5 sheet with your own handwritten notes (stamped by the Department of Mathematical Sciences), *Tabeller og formler i statistikk* (Tapir forlag), *Matematisk formelsamling* (K. Rottmann), calculator HP 30s or Citizen SR-270X

Grades to be announced: 24 December 2010

In the grading each of the ten points counts equally.

You should demonstrate how you arrive at your answers (e.g. by including intermediate answers or by referring to theory or examples from the reading list).

Throughout the problem set we assume that $X_1, X_2, \ldots, X_n$ are independent random variables from a distribution having probability density function f given by $f(x) = \theta/(1+x)^{\theta+1}$ for $x \geq 0$ and $f(x) = 0$ otherwise, where $\theta > 1$ is a parameter.

Problem 1

a) Show that the method of moments estimator of θ is $\tilde{\theta} = 1 + \dfrac{1}{\bar{X}}$.

b) Show that the maximum likelihood estimator of θ is $\hat{\theta} = \dfrac{n}{\sum_{i=1}^{n} \ln(1 + X_i)}$.

c) Show that the Cramér–Rao lower bound of the variance of an unbiased estimator of θ is θ^2/n.

d) Show that $E\hat{\theta} = \dfrac{n\theta}{n-1}$ and $\operatorname{Var}\hat{\theta} = \dfrac{n^2\theta^2}{(n-2)(n-1)^2}$.

(Hint: Show that $\ln(1 + X_i)$ is exponentially distributed, so that $\hat{\theta} = n/Z$, where Z is gamma distributed.)

e) Show that the family of probability density functions f is an exponential family. Show that $\sum_{i=1}^{n} \ln(1 + X_i)$ is a complete sufficient statistic for θ.

f) Find the unique best (uniform minimal variance) unbiased estimator of θ.

g) What is the asymptotic relative efficiency of $\tilde{\theta}$ with respect to $\hat{\theta}$ for $\theta > 2$? Find the numerical value when $\theta = 3$.

Problem 2

a) Find an approximate $1 - \alpha$ confidence interval for θ based on the maximum likelihood estimator $\hat{\theta}$. What is its realized numerical value if $n = 10$, $\hat{\theta} = 3.23$, and $\alpha = 0.05$?

b) Also find an exact $1 - \alpha$ confidence interval. (Hint: $2n\theta/\hat{\theta} \sim \chi^2_{2n}$.) What is its realized numerical value if $n = 10$, $\hat{\theta} = 3.23$, and $\alpha = 0.05$?

Problem 3

Find the uniformly most powerful level α test for testing $H_0\colon \theta \leq \theta_0$ versus $H_1\colon \theta > \theta_0$. For which values of $\hat{\theta}$ will H_0 be rejected if $\theta_0 = 3$, $n = 10$, and $\alpha = 0.05$?