Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4295 Stastistical inference

Friday 3 December 2010 9:00–13:00

Solutions

(Corrected 20 December 2010)

Problem 1

a)

$$EX_i = \int_0^\infty \frac{\theta x}{(1+x)^{\theta+1}}\,dx$$
$$= \theta \int_1^\infty \frac{u-1}{u^{\theta+1}}\,du = \theta \int_1^\infty (u^{-\theta} - u^{-\theta-1})\,du = \theta \left(\frac{1}{\theta-1} - \frac{1}{\theta}\right) = \frac{1}{\theta-1}.$$

The estimator $\tilde\theta$ is given by $\bar X = 1/(\tilde\theta - 1)$, that is, $\tilde\theta = 1 + 1/\bar X$.

b) The likelihood function is given by $L(\theta) = \theta^n / \prod_{i=1}^n (1+x_i)^{\theta+1}$, $\ln L(\theta) = n\ln\theta - (\theta+1)\sum_{i=1}^n \ln(1+x_i)$, $\partial \ln L(\theta)/\partial\theta = n/\theta - \sum_{i=1}^n \ln(1+x_i)$, $\partial^2 \ln L(\theta)/\partial\theta^2 = -n/\theta^2 < 0$, so that the MLE $\hat\theta$ is given by $n/\hat\theta - \sum_{i=1}^n \ln(1+X_i) = 0$, $\hat\theta = n/\sum_{i=1}^n \ln(1+X_i)$.

c) $\ln f(x) = \ln\theta - (\theta+1)\ln(1+x)$, $\partial \ln f(x)/\partial\theta = 1/\theta - \ln(1+x)$, $\partial^2 f(x)/\partial\theta^2 = -1/\theta^2$, so the Cramér–Rao bound is $1/(-nE\partial^2 f(X_i)/\partial\theta^2) = \theta^2/n$.

d) $P(\ln(1+X_i) \le y) = P(X_i \le e^y - 1)$, so that the pdf of $\ln(1+X_i)$ is given by

$$f(e^y-1)\cdot\frac{d}{dy}(e^y-1) = \frac{\theta}{(e^y)^{\theta+1}}\cdot e^y = \theta e^{-\theta y}$$

for $y \geq 0$. So $\hat{\theta} = n/Z$, where Z is a sum of n iid exponential variables, so that Z is gamma distributed having pdf given by $\frac{\theta^n}{\Gamma(n)} z^{n-1} e^{-\theta z}$.

$$E\hat{\theta} = \int_0^\infty \frac{n}{z} \frac{\theta^n}{\Gamma(n)} z^{n-1} e^{-\theta z} dz = \frac{n\theta}{n-1} \int_0^\infty \frac{\theta^{n-1}}{\Gamma(n-1)} z^{n-2} e^{-\theta z} dz = \frac{n\theta}{n-1}$$

and

$$E\hat{\theta}^2 = \int_0^\infty \frac{n^2}{z^2} \frac{\theta^n}{\Gamma(n)} z^{n-1} e^{-\theta z} dz = \frac{n^2\theta^2}{(n-1)(n-2)} \int_0^\infty \frac{\theta^{n-2}}{\Gamma(n-2)} z^{n-3} e^{-\theta z} dz = \frac{n^2\theta^2}{(n-1)(n-2)}$$

(we recognize the integrands at the right as gamma densities), and

$$\operatorname{Var}\hat{\theta} = E\hat{\theta}^2 - (E\hat{\theta})^2 = \frac{n^2\theta^2}{n-1}\left(\frac{1}{n-2} - \frac{1}{n-1}\right) = \frac{n^2\theta^2}{(n-2)(n-1)^2}.$$

e)

$$f(x) = \frac{\theta}{(1+x)^{\theta+1}} = \theta e^{-(\theta+1)\ln(1+x)},$$

which is of the required form, and $\sum_{i=1}^n \ln(1+X_i)$ is complete and sufficient for θ.

f)

$$\frac{n-1}{n}\hat{\theta} = \frac{n-1}{\sum_{i=1}^n \ln(1+X_i)}$$

is unbiased and a function of a complete sufficient statistic for θ, and thus the UMVUE.

g) By asymptotic properties of MLEs, $\sqrt{n}(\hat{\theta} - \theta) \to N(0, \theta^2)$ in distribution as $n \to \infty$. We need the corresponding description of $\tilde{\theta}$, and start by finding an asymptotic description of $\bar{X}$.

First,

$$\begin{aligned} EX_i^2 &= \int_0^\infty \frac{\theta x^2}{(1+x)^{\theta+1}}\, dx = \theta \int_1^\infty \frac{(u-1)^2}{u^{\theta+1}}\, du \\ &= \theta \int_1^\infty (u^{-\theta+1} - 2u^{-\theta} + u^{-\theta-1})\, du = \theta\left(\frac{1}{\theta-2} - \frac{2}{\theta-1} + \frac{1}{\theta}\right) = \frac{2}{(\theta-2)(\theta-1)}, \end{aligned}$$

so that

$$\operatorname{Var} X_i = EX_i^2 - (EX_i)^2 = \frac{2}{(\theta-2)(\theta-1)} - \frac{1}{(\theta-1)^2} = \frac{\theta}{(\theta-2)(\theta-1)^2}.$$

By the central limit theorem,

$$\sqrt{n}\left(\bar{X} - \frac{1}{\theta-1}\right) \to N\left(0, \frac{\theta}{(\theta-2)(\theta-1)^2}\right)$$

in distribution. Applying the Delta method using $g(t) = 1 + 1/t$, so that $g'(t) = -1/t^2$,

$$g(\bar{X}) = \tilde{\theta}, \qquad g\left(\frac{1}{\theta - 1}\right) = \theta, \qquad g'\left(\frac{1}{\theta-1}\right) = -(\theta-1)^2,$$

$$\frac{\theta}{(\theta-2)(\theta-1)^2}\left(g'\left(\frac{1}{\theta-1}\right)\right)^2 = \frac{\theta(\theta-1)^2}{\theta-2},$$

yielding

$$\sqrt{n}(\tilde{\theta} - \theta) \to N\left(0, \frac{\theta(\theta-1)^2}{\theta-2}\right),$$

and the ARE of $\tilde{\theta}$ with respect to $\hat{\theta}$ is

$$\frac{\theta^2}{\theta(\theta-1)^2/(\theta-2)} = \frac{(\theta-2)\theta}{(\theta-1)^2},$$

which is 0.75 for $\theta = 3$.

Problem 2

a)

$$1 - \alpha \approx P\left(-z_{\alpha/2} \le \frac{\hat{\theta} - \theta}{\theta/\sqrt{n}} \le z_{\alpha/2}\right) = P\left(-z_{\alpha/2} \le \sqrt{n}\left(\frac{\hat{\theta}}{\theta} - 1\right) \le z_{\alpha/2}\right).$$

Solving the inequalities with respect to θ yields

$$P\left(\frac{\hat{\theta}}{1 + z_{\alpha/2}/\sqrt{n}} \le \theta \le \frac{\hat{\theta}}{1 - z_{\alpha/2}/\sqrt{n}}\right) \approx 1 - \alpha.$$

The realized interval is $[1.99, 8.50]$.

If we instead replace θ by $\hat{\theta}$ for the variance, we get

$$P\left(\left(1 - \frac{z_{\alpha/2}}{\sqrt{n}}\right)\hat{\theta} \le \theta \le \left(1 + \frac{z_{\alpha/2}}{\sqrt{n}}\right)\hat{\theta}\right) \approx 1 - \alpha$$

and a realized interval of $[1.23, 5.23]$.

b) Since $n/\hat{\theta} \sim \text{Gamma}(n, 1/\theta)$, $2n\theta/\hat{\theta} \sim \text{Gamma}(n, 2) = \chi^2_{2n}$. So

$$1 - \alpha = P\left(\chi^2_{2n,1-\alpha/2} \le \frac{2n\theta}{\hat{\theta}} \le \chi^2_{2n,\alpha/2}\right) = P\left(\frac{\chi^2_{2n,1-\alpha/2}\hat{\theta}}{2n} \le \theta \le \frac{\chi^2_{2n,\alpha/2}\hat{\theta}}{2n}\right).$$

The realized interval is $[1.55, 5.52]$.

Problem 3

Assume $\theta_2 > \theta_1$. Let $T = \sum_{i=1}^n \ln(1+X_i)$ have pdf g. Then

$$\frac{\prod_{i=1}^n f(x_i;\theta_2)}{\prod_{i=1}^n f(x_i;\theta_1)} = \frac{\theta_2^n / \prod_{i=1}^n (1+x_i)^{\theta_2+1}}{\theta_1^n / \prod_{i=1}^n (1+x_i)^{\theta_1+1}}$$
$$= \left(\frac{\theta_2}{\theta_1}\right)^n \prod_{i=1}^n (1+x_i)^{\theta_1-\theta_2} = \left(\frac{\theta_2}{\theta_1}\right)^n \exp\left((\theta_1-\theta_2)\sum_{i=1}^n \ln(1+x_i)\right),$$

so that $g(t;\theta_2)/g(t;\theta_1) = (\theta_2/\theta_1)^n e^{(\theta_1-\theta_2)t}$, which is a decreasing function of t. By Karlin–Rubin's Theorem, the test that rejects H_0 for $T < t_0$ is a UMP level $P_{\theta_0}(T < t_0)$ test.

We want $P_{\theta_0}(T < t_0) = \alpha$. Since $T = n/\hat{\theta}$ and $2n\theta/\hat{\theta} \sim \chi^2_{2n}$, this translates to $2n\theta_0/\hat{\theta} < \chi^2_{2n,1-\alpha}$, that is, $\hat{\theta} > 2n\theta_0/\chi^2_{2n,1-\alpha}$. With $\theta_0 = 3,\ n = 10,\ \alpha = 0.05$, we reject H_0 if $\hat{\theta} > 2\cdot 10\cdot 3/10.85 \approx 5.53$.